# Exercises 04.02.

Find and document mistakes in the lecture notes.

Find the parameters for a laser pulse that implements a resonant Hadamard gate

How many rotations by an angle $\beta = 1$ do you need for an approximate rotation by $\alpha = \pi/2$ if the error should be < 0.1 ? How many for $< 10^{-2}$ ? How many for $< 10^{-3}$ ?

Find suitable parameters for implementing a Hadamard gate with a near-resonant ac field for (i) a spin with a Larmor frequency of 100 MHz and (ii) for an atom with a energy level splitting of 2 eV.

# Exercises QIP 02/4/26

1. **Qubit Projector in the Pauli Basis**
   Given a general single-qubit state:
   $$|\psi\rangle = a\,|0\rangle + b\,|1\rangle$$
   where $a$ and $b$ are complex numbers satisfying $|a|^2 + |b|^2 = 1$, express the projector $|\psi\rangle\langle\psi|$ as a linear combination of Pauli matrices:
   $$|\psi\rangle\langle\psi| = c_1 I + c_x X + c_y Y + c_z Z$$

2. **Decoherence: Pure Dephasing from a Stochastic Phase**
   Consider a spin-$\frac{1}{2}$ particle initially polarized along the $x$-axis, i.e.,
   $$\rho(0) = \frac{1}{2}I + S_x,$$
   assuming $\hbar = 1$, and subject to a time-dependent stochastic phase $\delta(t)$ accumulated due to a fluctuating magnetic field along the $z$-axis. That is, $\delta(t)$ incorporates the effect of the field over time, and represents the net phase difference acquired between the spin states $|0\rangle$ and $|1\rangle$. In the interaction (rotating) frame, the time evolution of the density matrix is given by
   $$\rho(t) = e^{-i\delta(t)S_z}\,\rho(0)\,e^{i\delta(t)S_z}.$$
   We are interested in computing the normalized ensemble-averaged magnetization along the $x$-axis,
   $$\langle S_x(t)\rangle = \frac{\mathrm{Tr}\,[\rho(t)S_x]}{\mathrm{Tr}\,[\rho(0)S_x]}.$$

   (a) Show that
   $$\langle S_x(t)\rangle = \langle\cos(\delta(t))\rangle,$$
   assuming that $\delta(t)$ is a real-valued stochastic variable with zero mean.

   (b) Suppose $\delta(t)$ is a *Gaussian random variable* with zero mean and variance $\langle\delta^2(t)\rangle$. Express $\langle S_x(t)\rangle$ in terms of the real part of the characteristic function $\langle e^{i\delta(t)}\rangle$.

   (c) Prove that for a zero-mean Gaussian random variable $\delta$,
   $$\langle e^{i\delta}\rangle = e^{-\langle\delta^2\rangle/2}.$$
   *Hint:* Expand $e^{i\delta}$ in a Taylor series and use the known moments of a Gaussian variable: all odd moments vanish, and even moments are given by
   $$\langle\delta^{2n}\rangle = (2n-1)!!\,\langle\delta^2\rangle^n.$$

   (d) Use the result from part (c) to show that
   $$\langle S_x(t)\rangle = e^{-\langle\delta^2(t)\rangle/2}.$$

   (e) Consider the case where $\delta(t)$ arises from a random walk process (Markovian noise), such that
   $$\langle\delta^2(t)\rangle = \frac{2t}{T_2},$$
   i.e., the phase variance grows linearly with time. Show that
   $$\langle S_x(t)\rangle = e^{-t/T_2},$$
   and interpret the physical meaning of the time constant $T_2$.