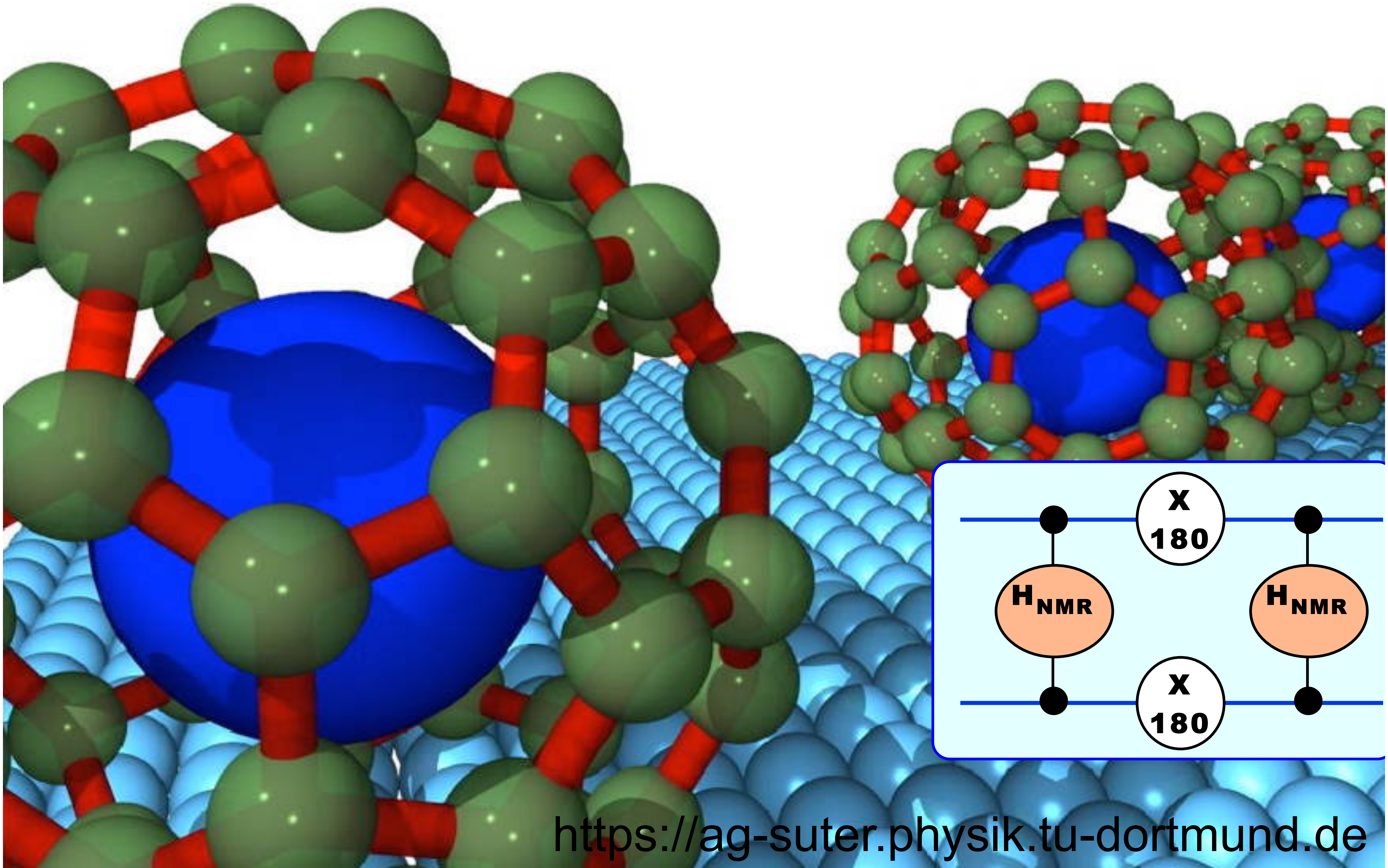
Quantum Computers



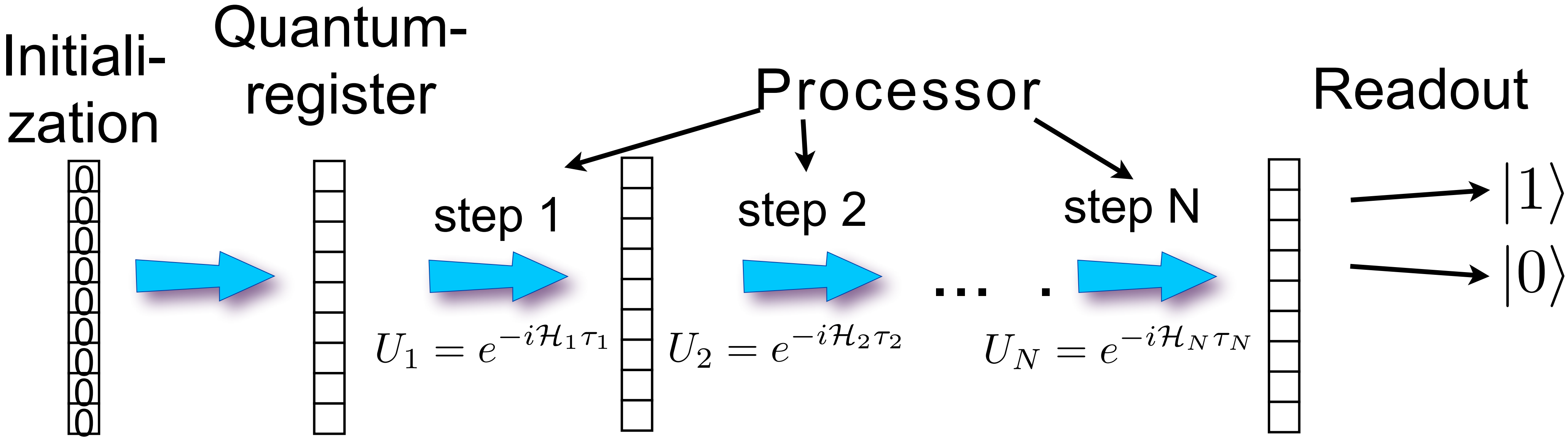
https://ag-suter.physik.tu-dortmund.de

Contents

7) Errors and Decoherence

Motivation

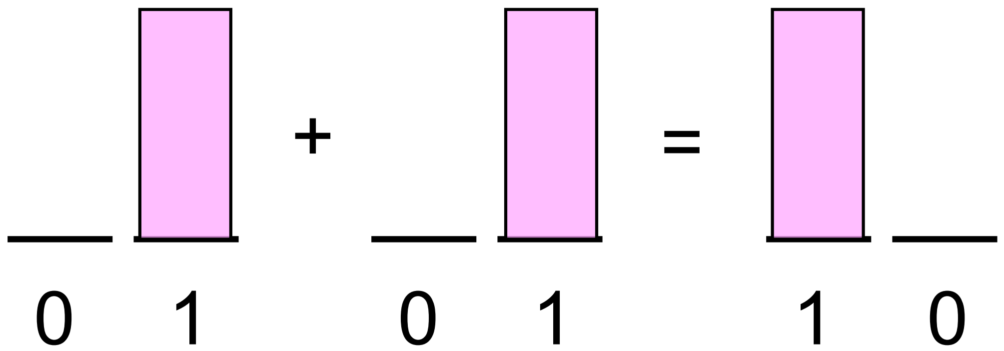
Motivation



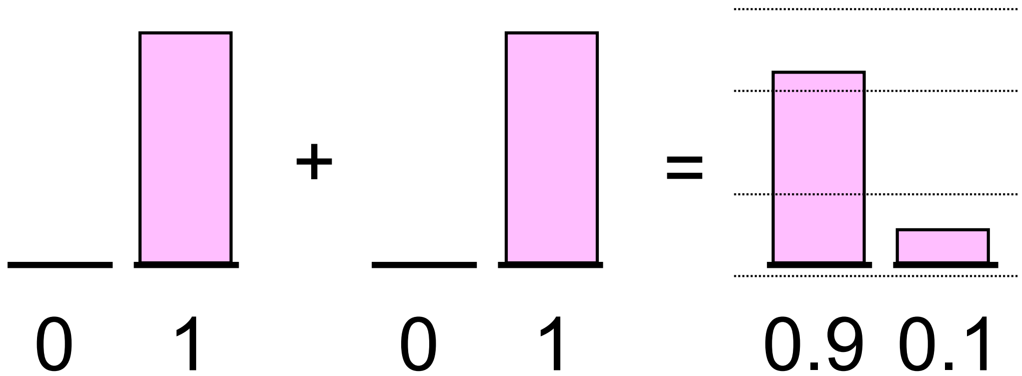
Physical systems behave differently

Classical Error Correction

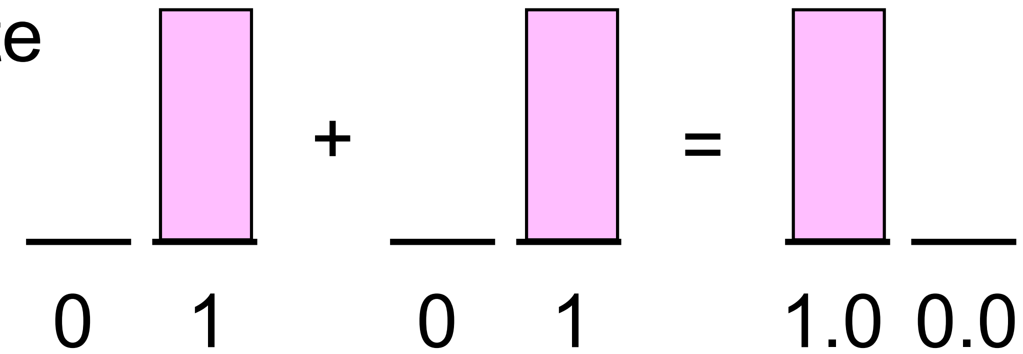
Ideal



Real



Recalibrate

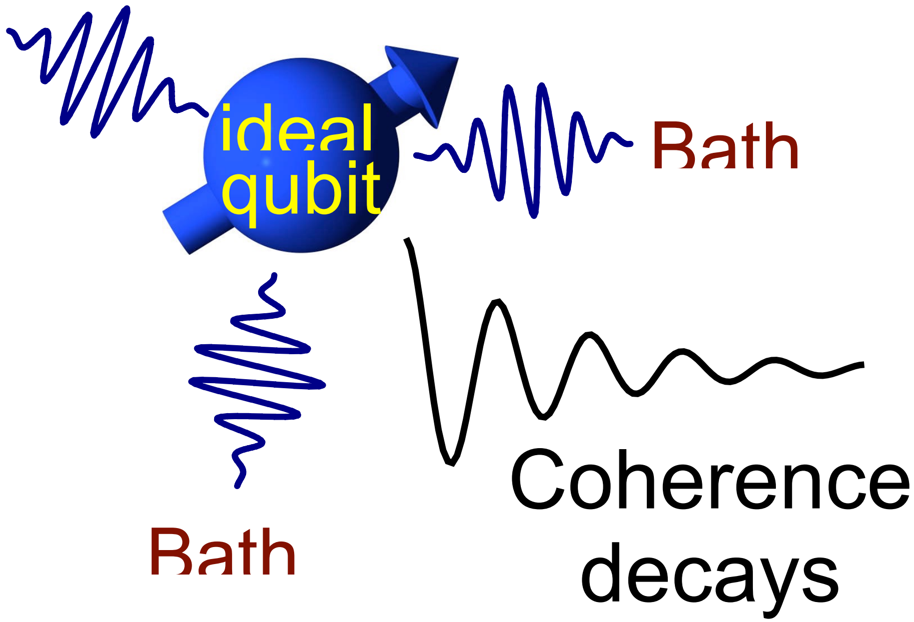


Sources of Errors

- Parameters of quantum register differ from the ideal ones

- Control fields have finite precision → errors

- Coupling to environment → decoherence



Fighting Errors

- Optimize the classical apparatus that controls the quantum system to make the gate operations as perfect as possible.
- Design gate operations in such a way that errors in experimental parameters tend to cancel rather than amplify.
- Use error correction schemes.
- Store the information in areas of the Hilbert space that are least affected by the interaction between the system and its environment.
- Use active schemes for decoupling the system from the environment, such as dynamical decoupling.

All schemes must be combined!

Literature

REVIEWS OF MODERN PHYSICS, VOLUME 88, OCTOBER–DECEMBER 2016

Colloquium: Protecting quantum information against environmental noise

Dieter Suter

Fakultät Physik, TU Dortmund, 44221 Dortmund, Germany

Gonzalo A. Álvarez

Weizmann Institute of Science, 76100 Rehovot, Israel,
and Centro Atómico Bariloche, CNEA, CONICET, 8400 S. C. de Bariloche, Argentina

(published 10 October 2016)

D. Suter and G. A. Álvarez,
"Protecting quantum information against environmental noise",
Rev. Mod. Phys. 88, 041001 (2016).

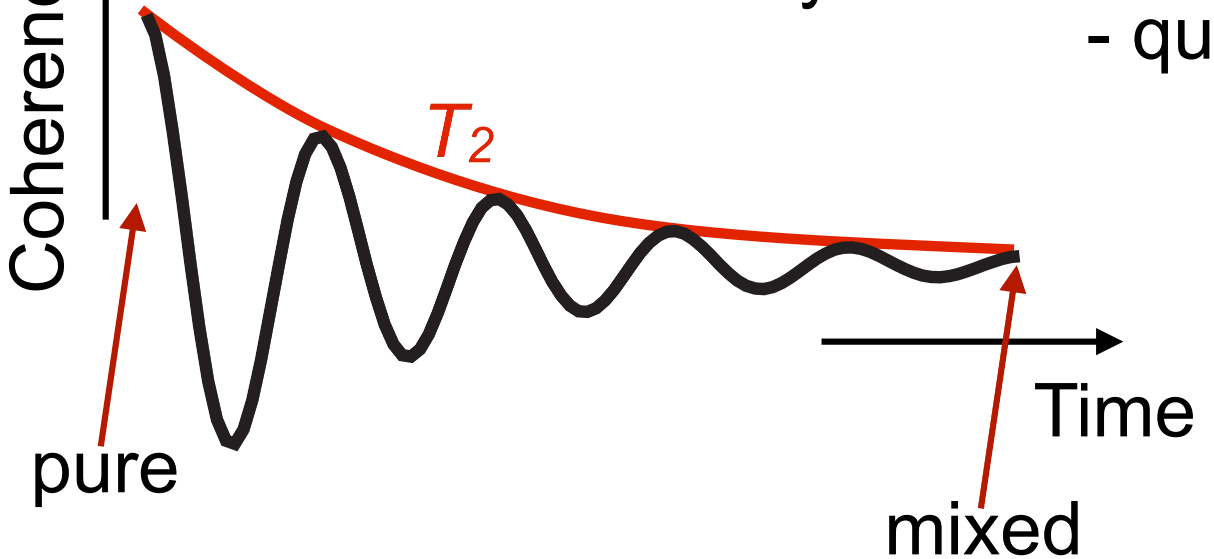
Phenomenology of Decoherence

Superposition state $|\Psi(0)\rangle = a|0\rangle + b|1\rangle$

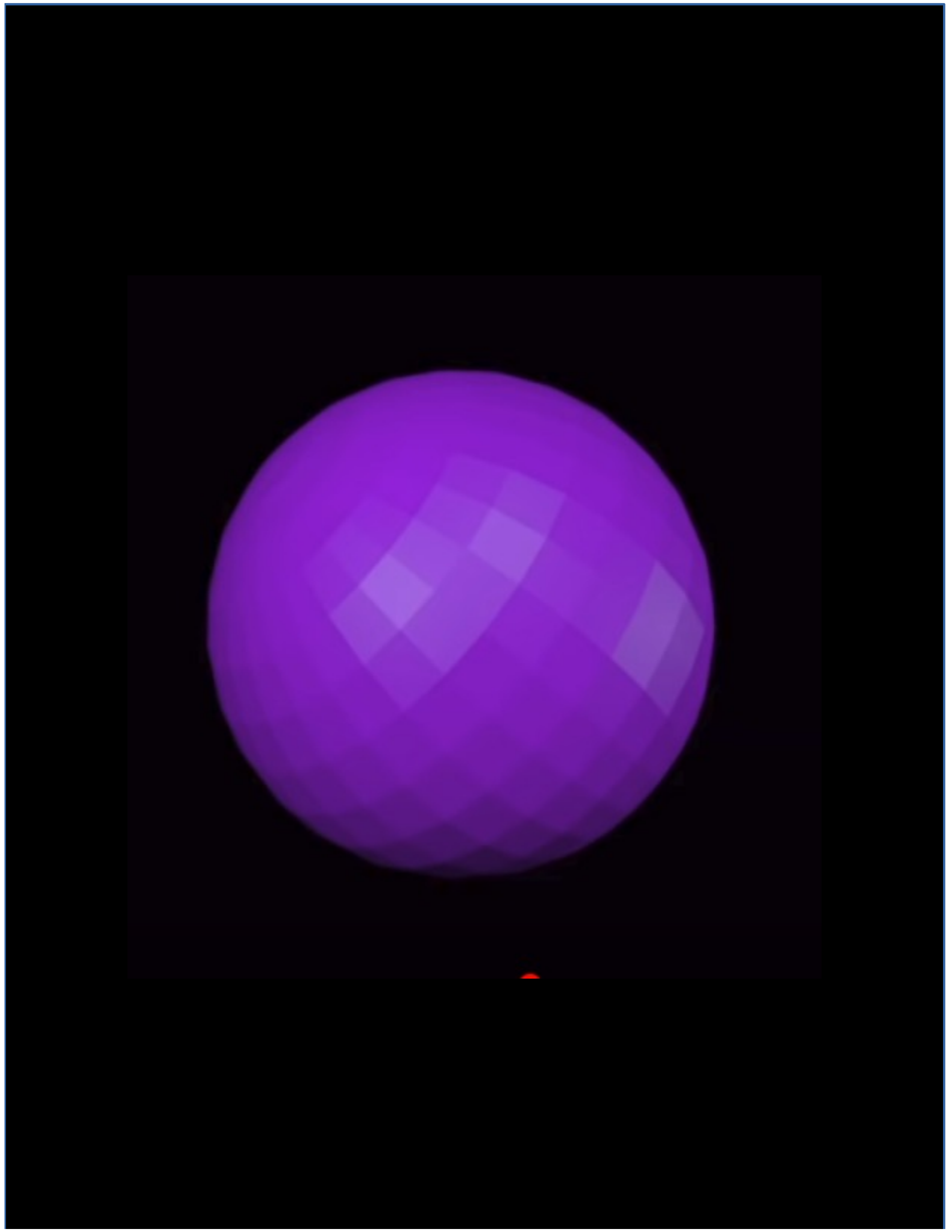
Coherence = a^*b

From chapter 4: if qubit gets entangled with the environment, its partial density operator becomes mixed

FID =
Free Induction Decay

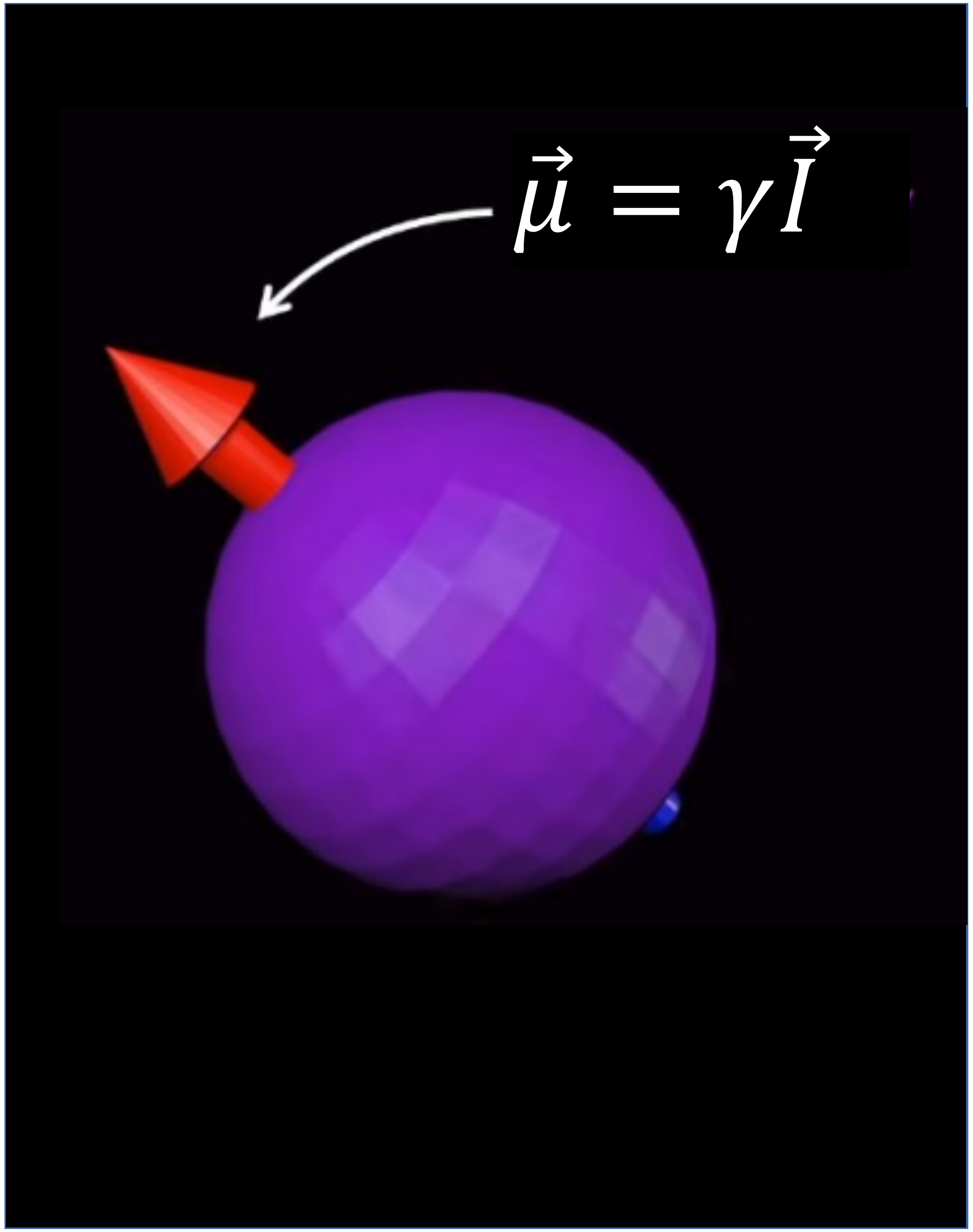


Description of environment
- classical : fluctuations
- quantum : entanglement



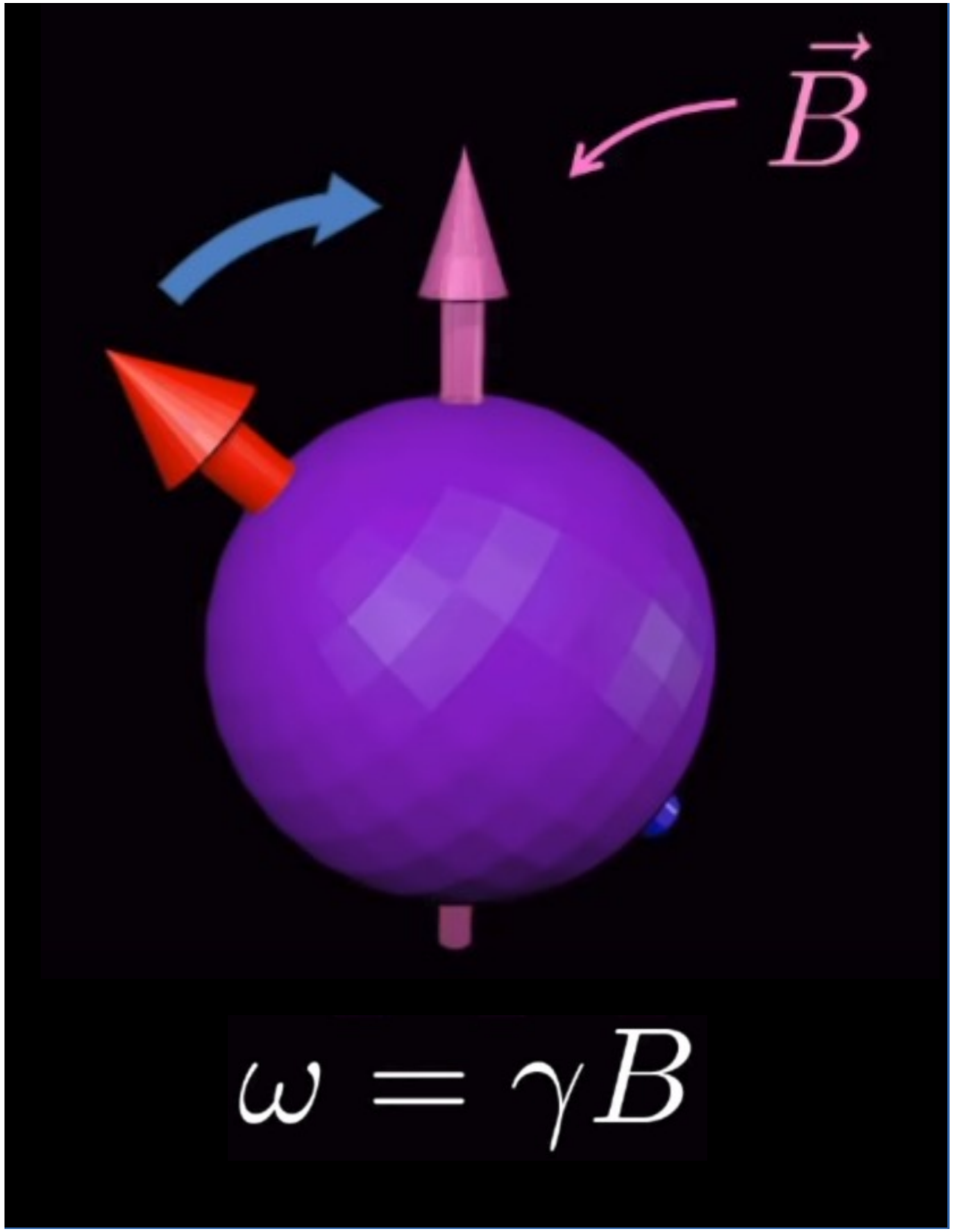
Proton 1H nuclei

- Mass
- Charge
- **SPIN** ***s*** =1/2



Proton 1H nuclei

- Mass
- Charge
- **SPIN** ***s*** =1/2
  - (i) Angular Momentum
  - (ii) Magnetic Moment



Proton 1H nuclei

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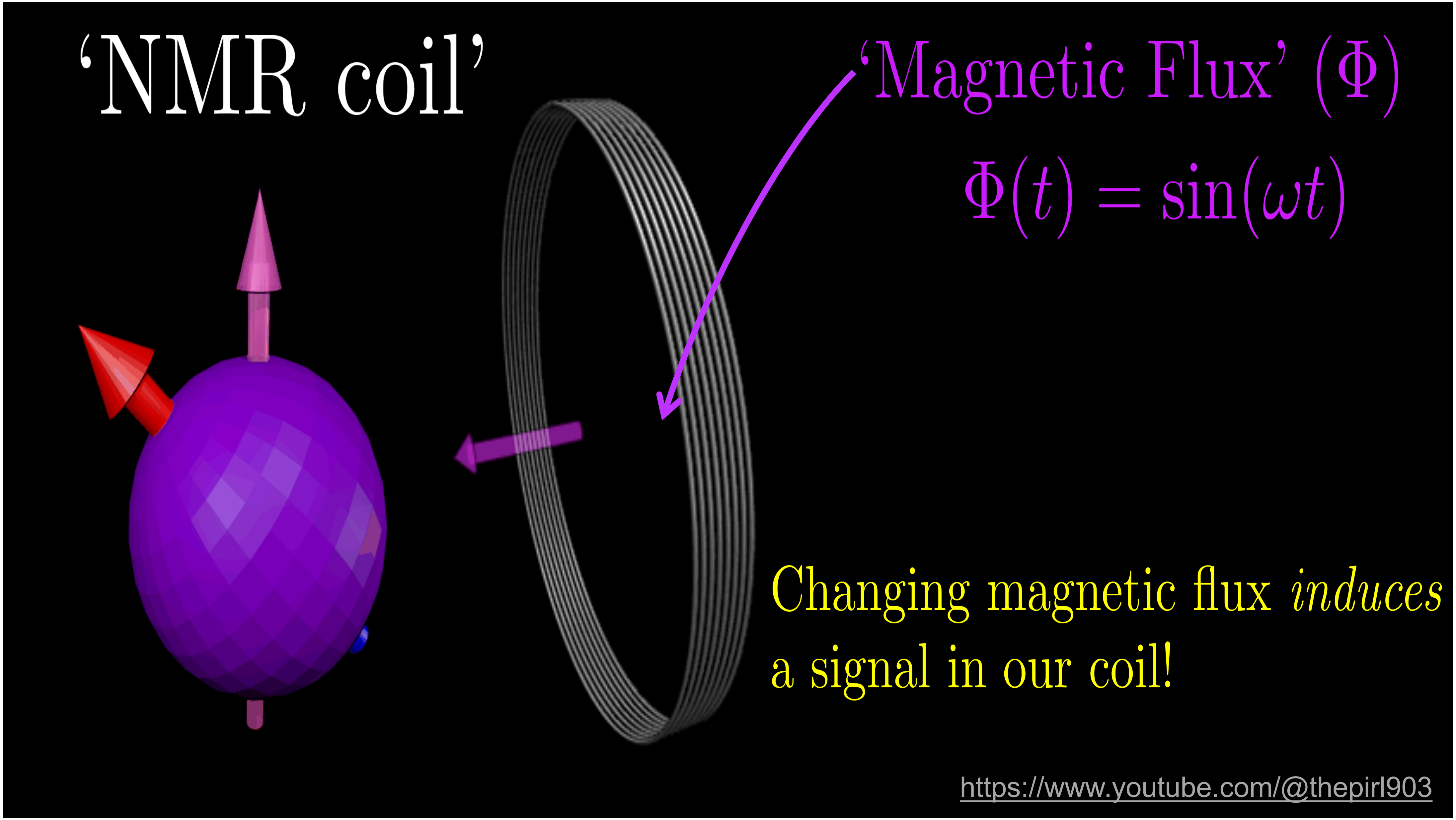
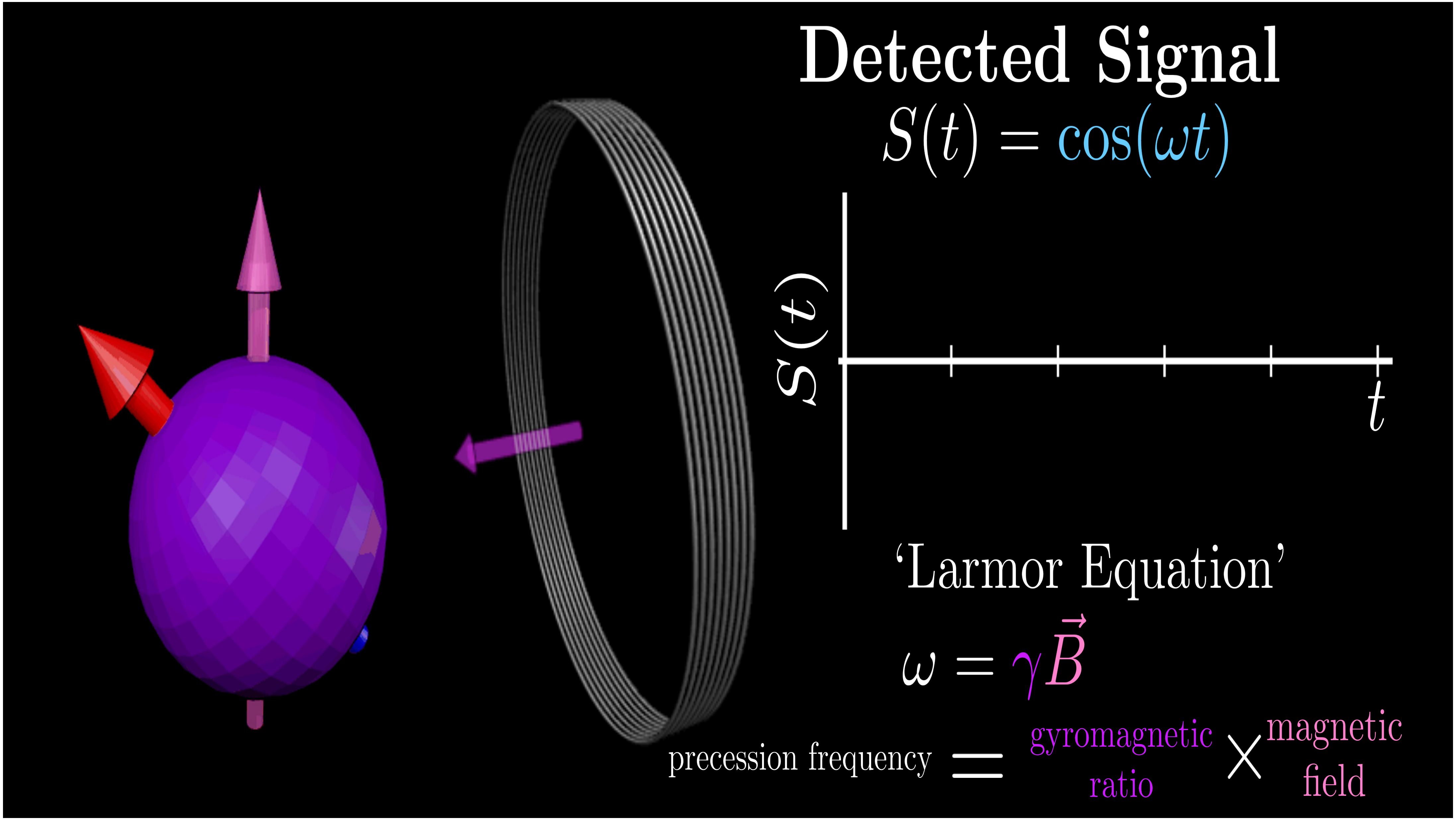
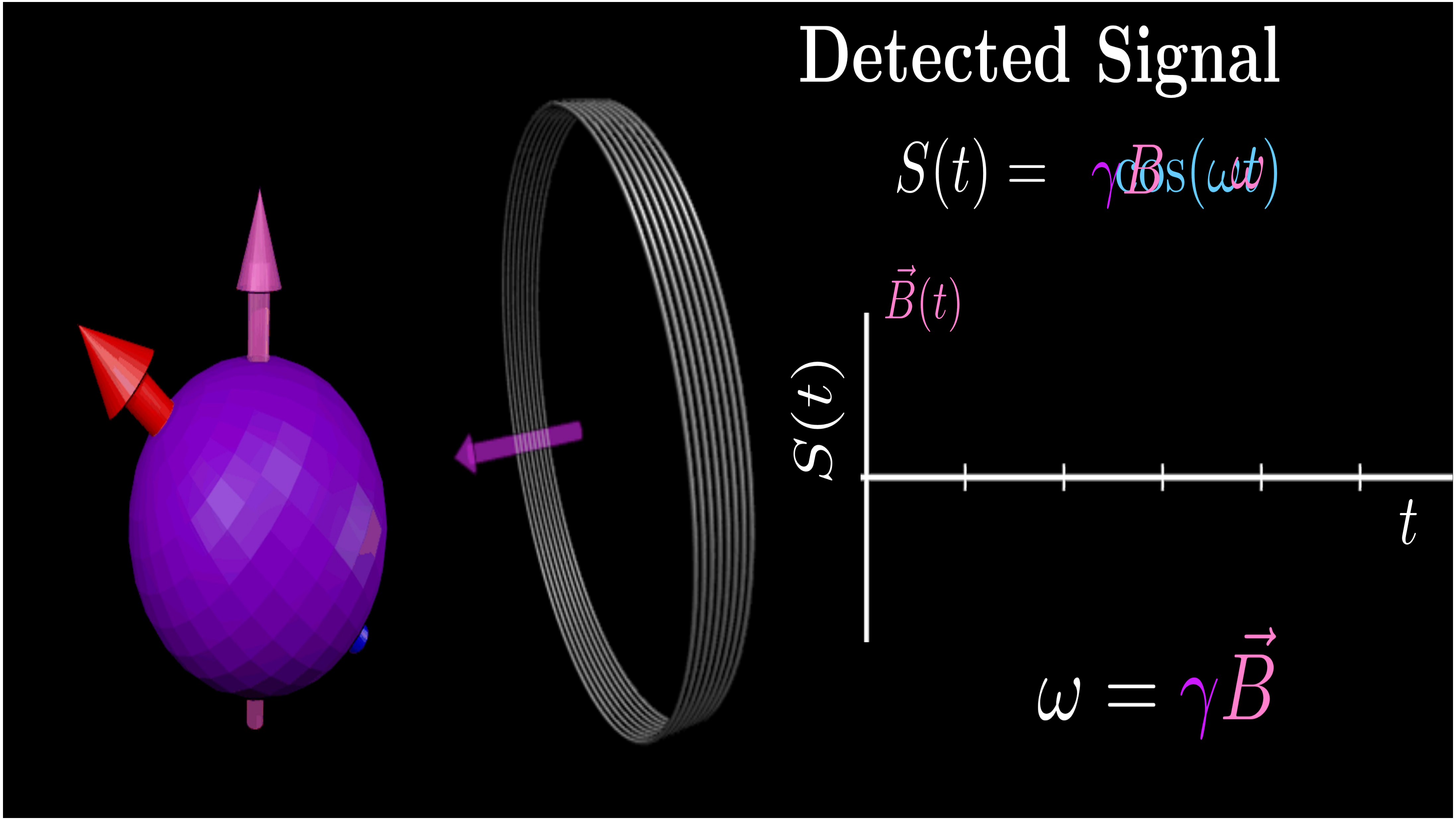
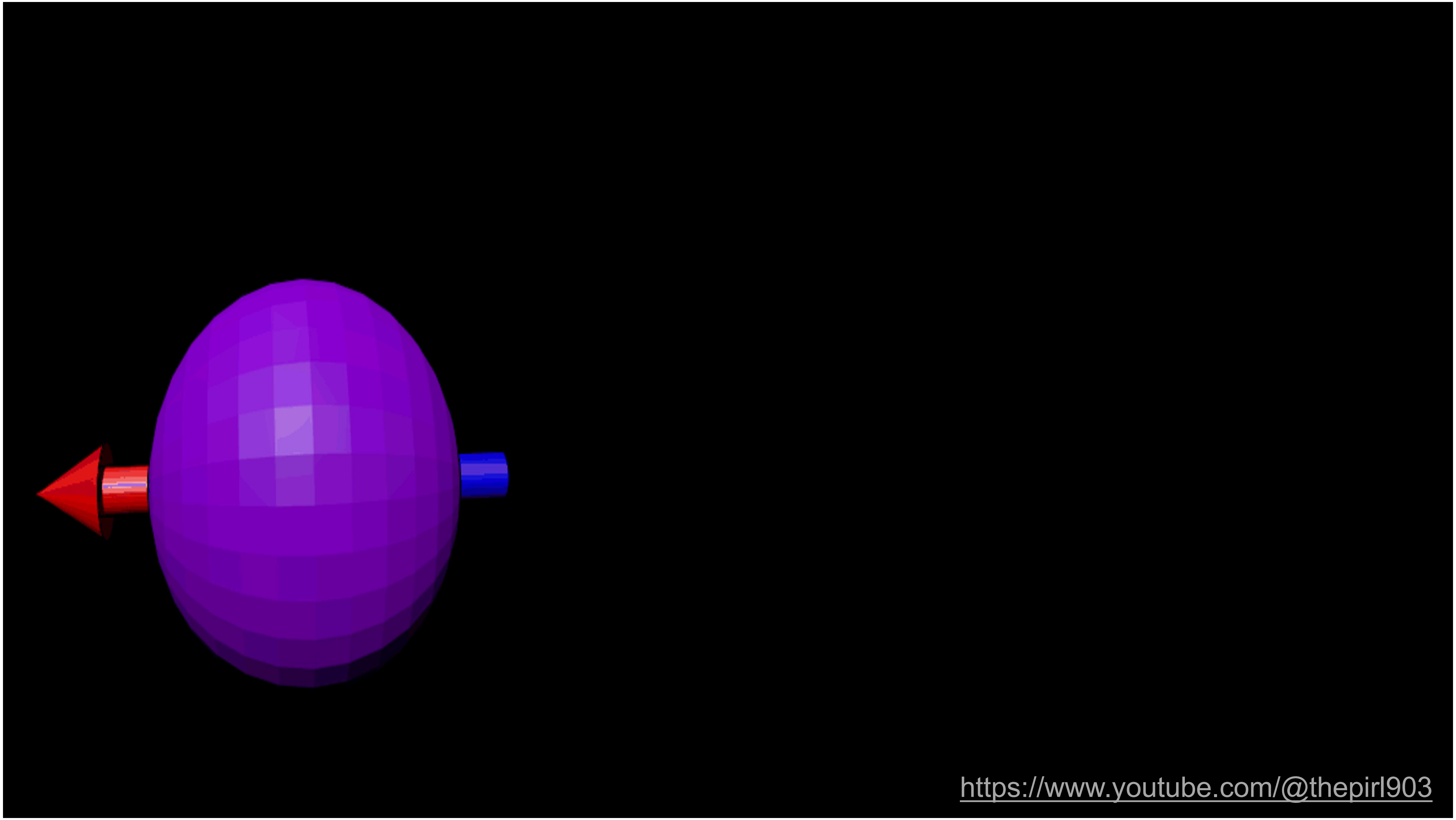

'NMR coil'
'Magnetic Flux' (Φ)
$\Phi(t) = \sin(\omega t)$
Changing magnetic flux *induces* a signal in our coil!
https://www.youtube.com/@thepirl903


Detected Signal
$S(t) = \cos(\omega t)$
$S(t)$
t
'Larmor Equation'
$\omega = \gamma \vec{B}$
precession frequency = gyromagnetic ratio × magnetic field


Detected Signal
$S(t) = \gamma B \cos(\omega t)$
$\vec{B}(t)$
$S(t)$
t
$\omega = \gamma \vec{B}$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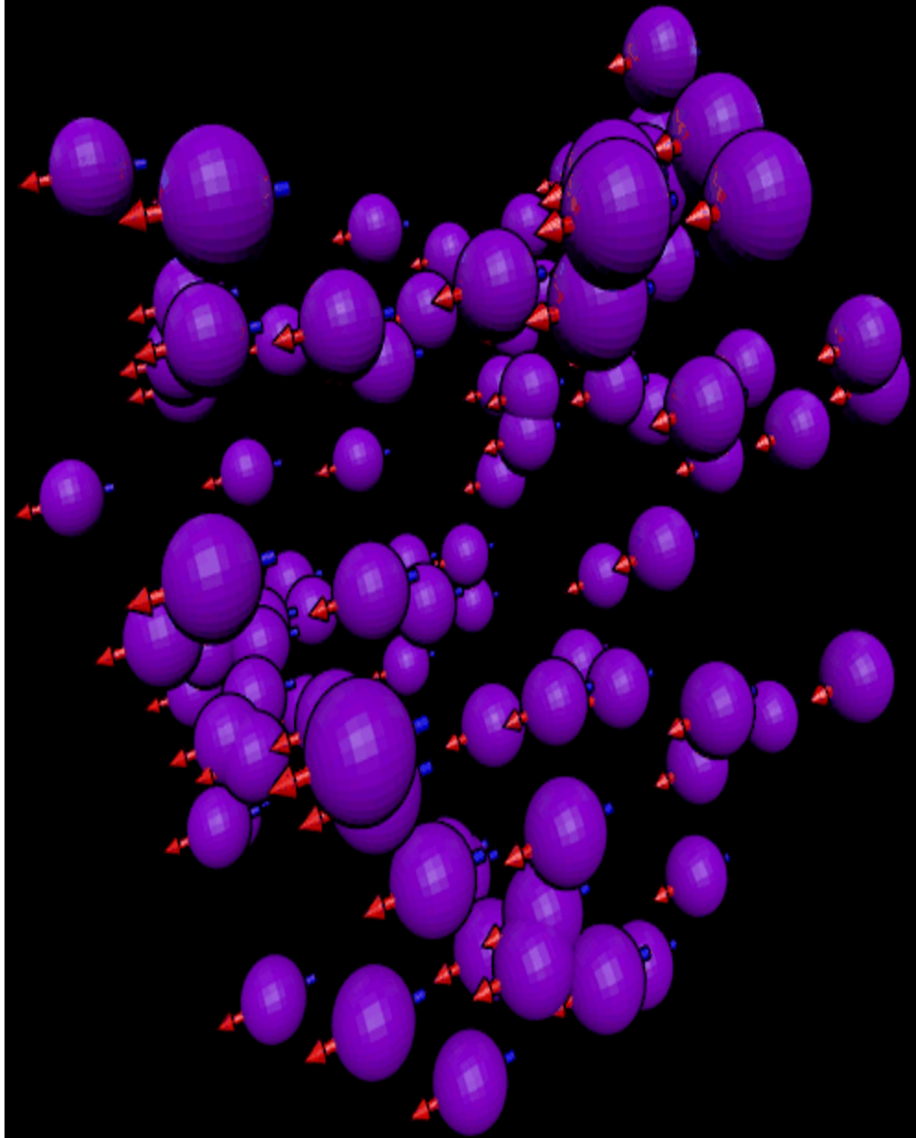


$$S(t) = N \sin(\theta)\gamma B \cos(\omega t)$$

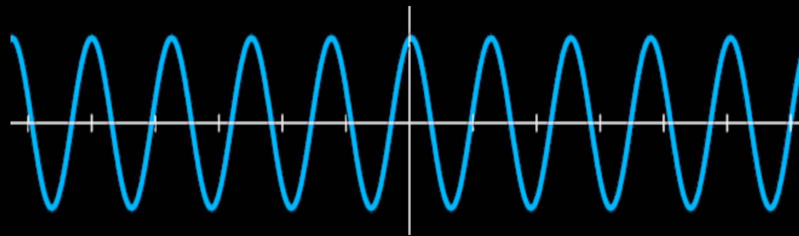
number of spins

flip angle

gyromagnetic ratio

magnetic field

$$S(t) = A\cos(\omega t)$$



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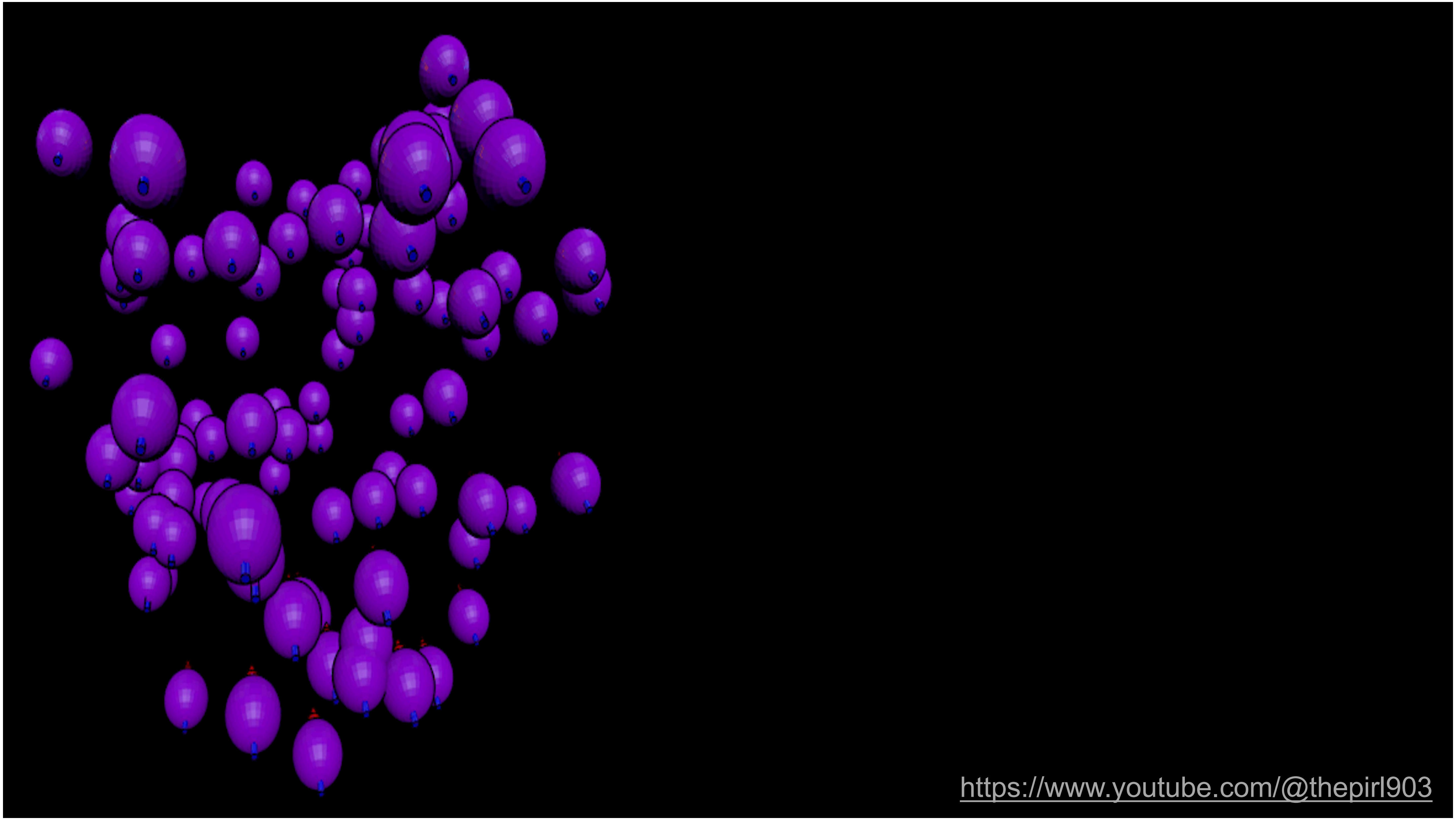
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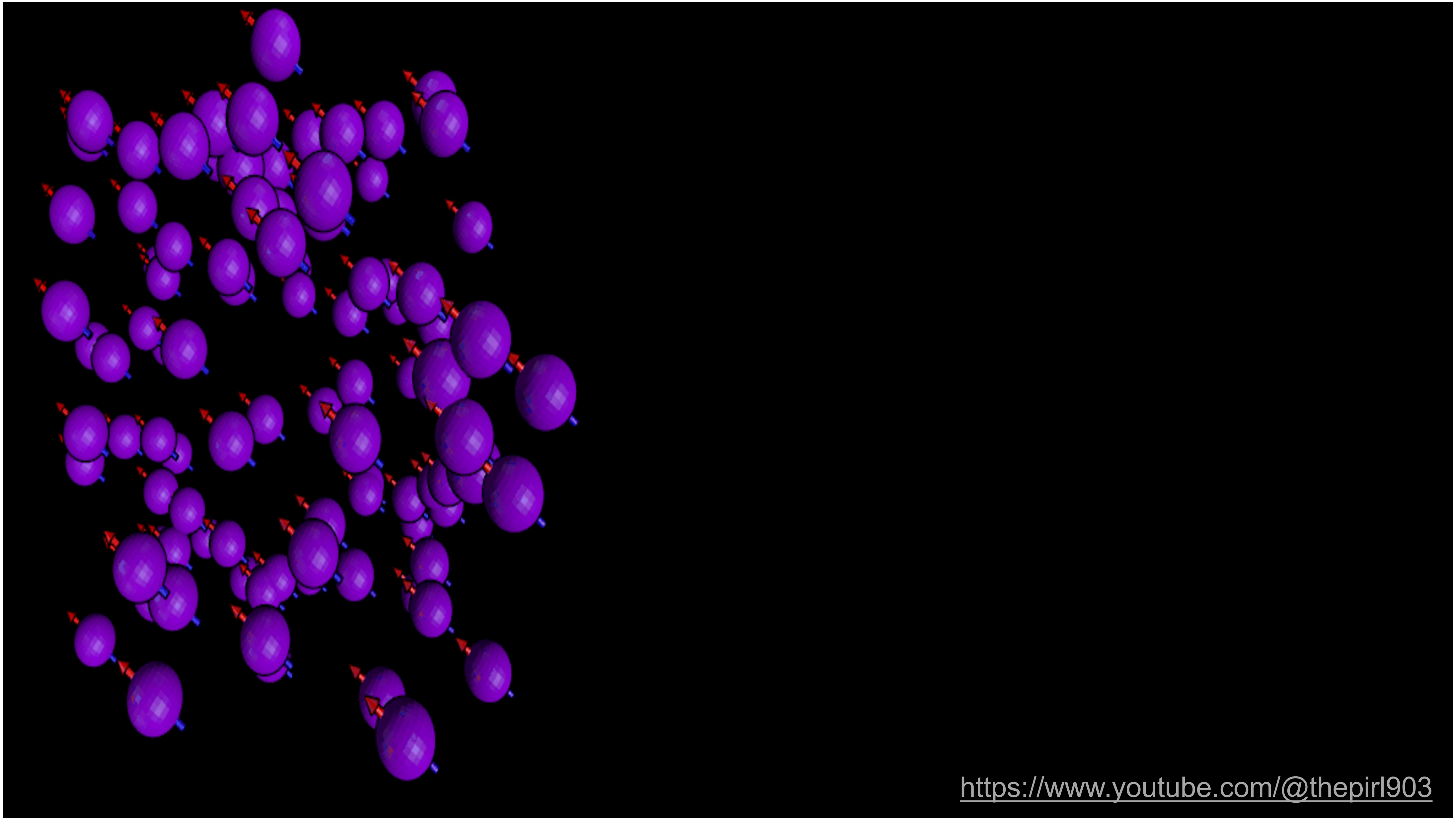
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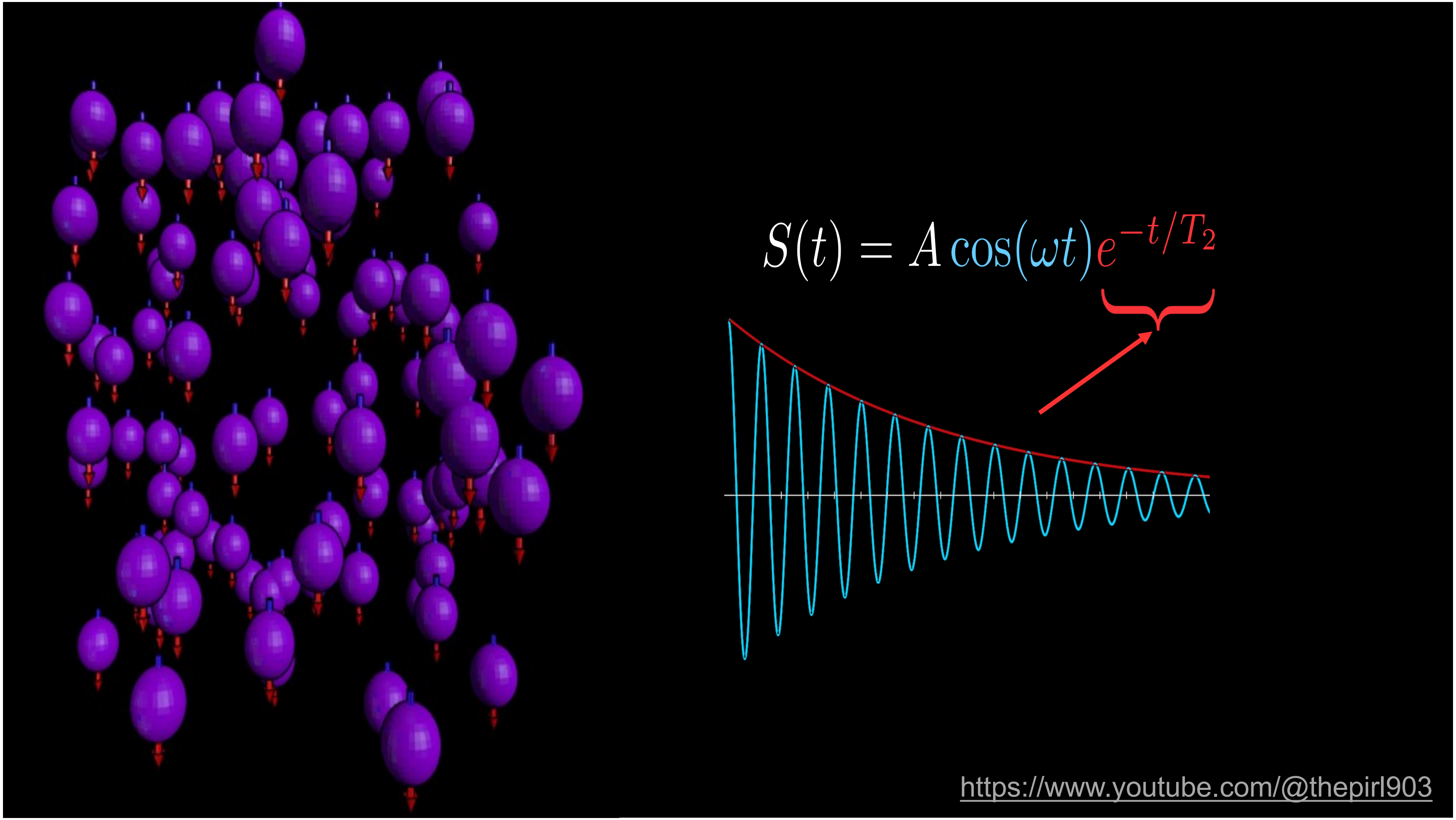
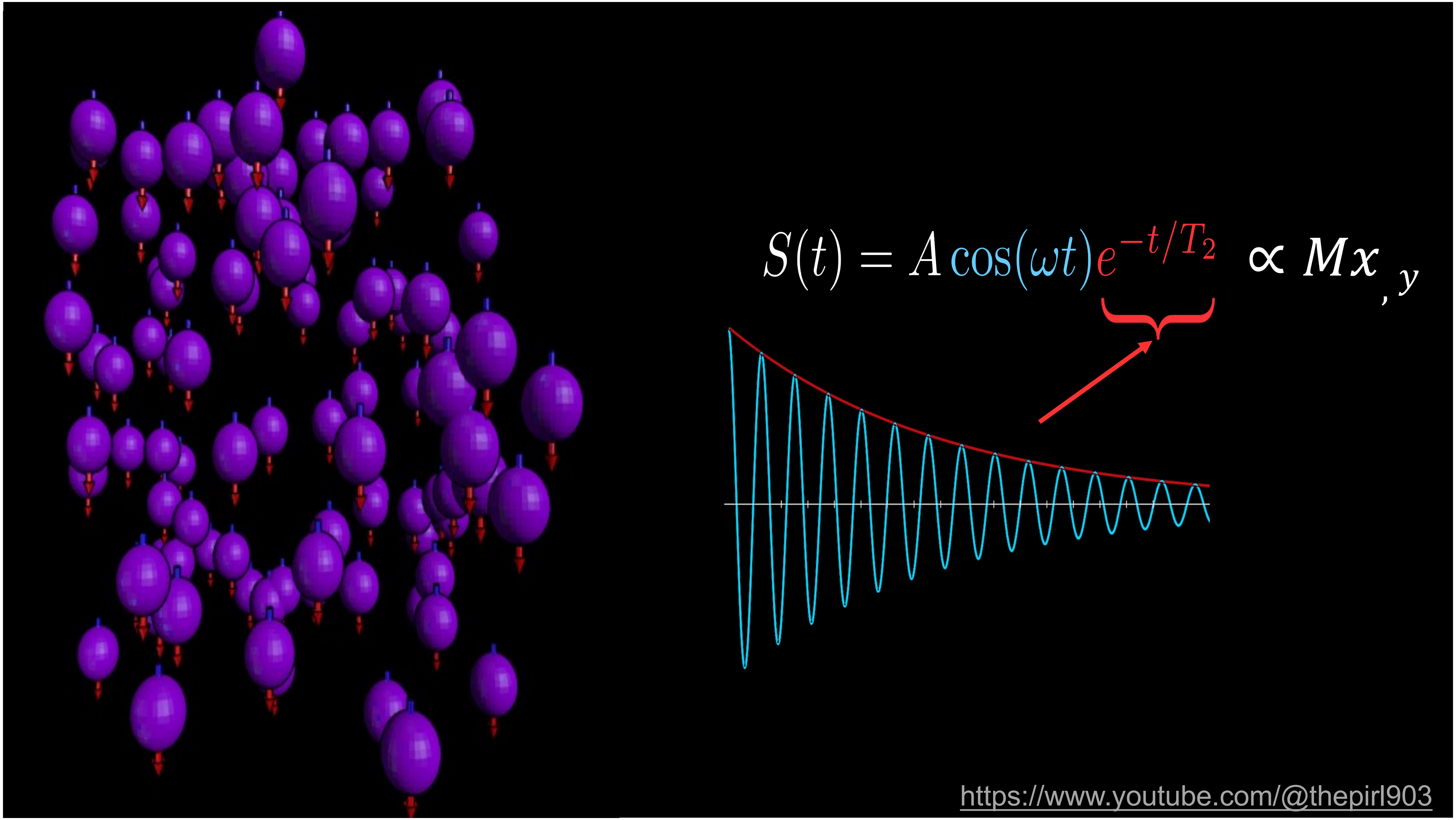
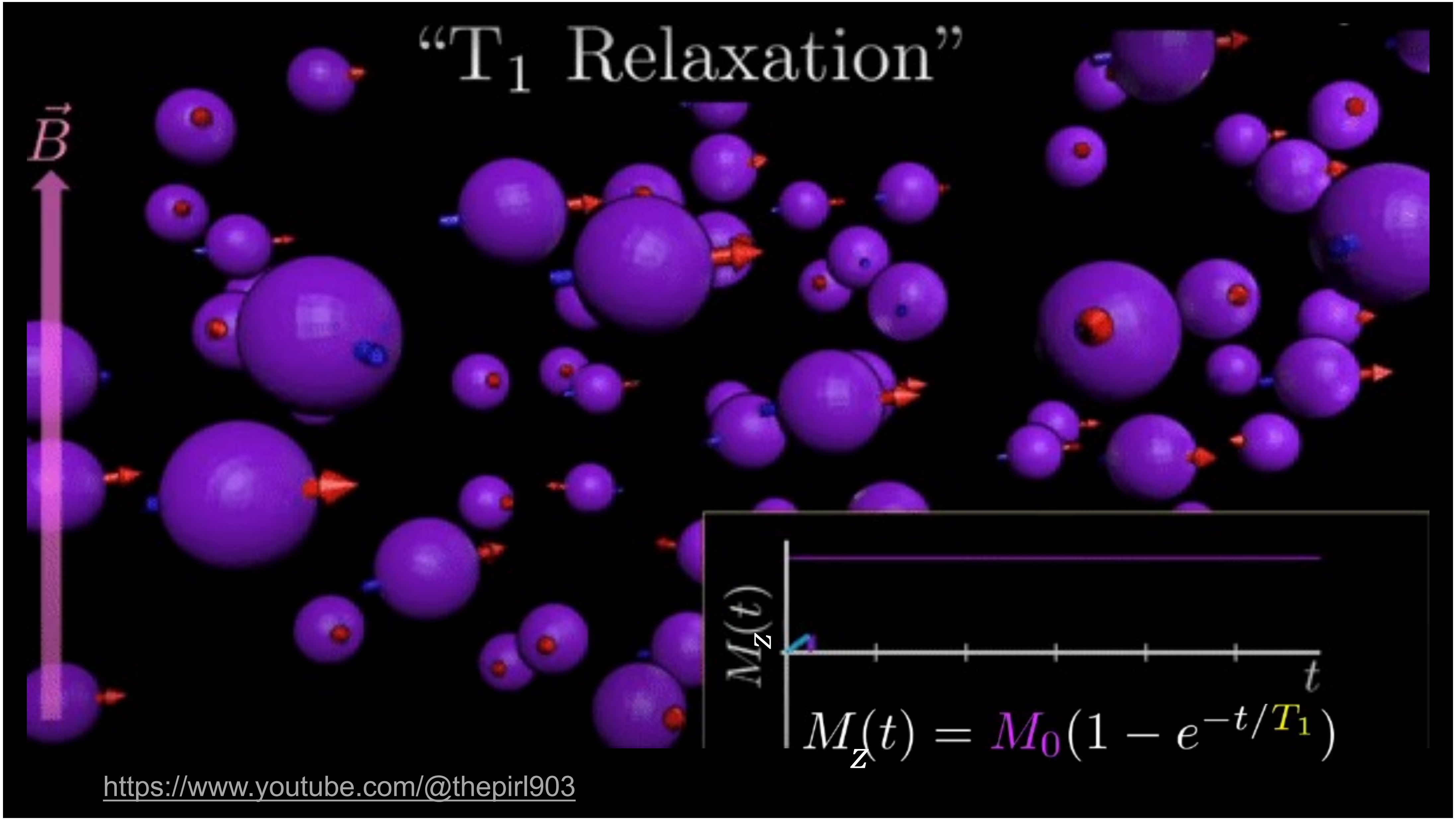


$S(t) = A\cos(\omega t)\underbrace{e^{-t/T_2}}$
https://www.youtube.com/@thepirl903

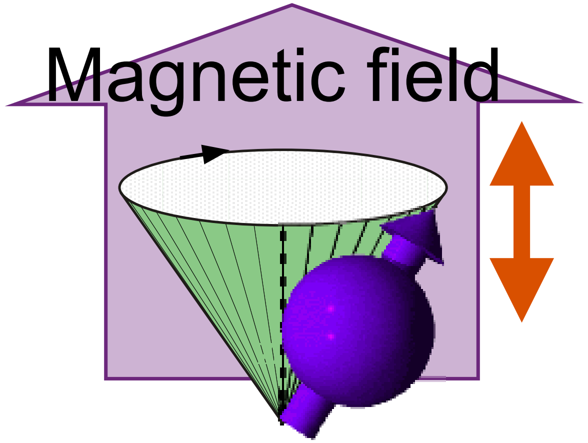

$S(t) = A\cos(\omega t)e^{-t/T_2} \propto Mx_{,y}$
https://www.youtube.com/@thepirl903

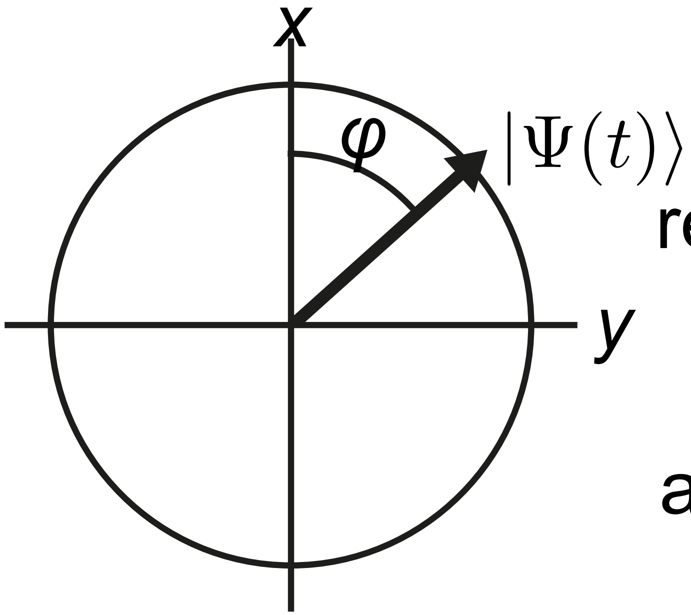

"T1 Relaxation"
$\vec{B}$
$M_z(t)$
t
$M_z(t) = M_0(1 - e^{-t/T_1})$
https://www.youtube.com/@thepirl903

Semiclassical Description

$$|\Psi(0)\rangle = a|0\rangle + b|1\rangle$$

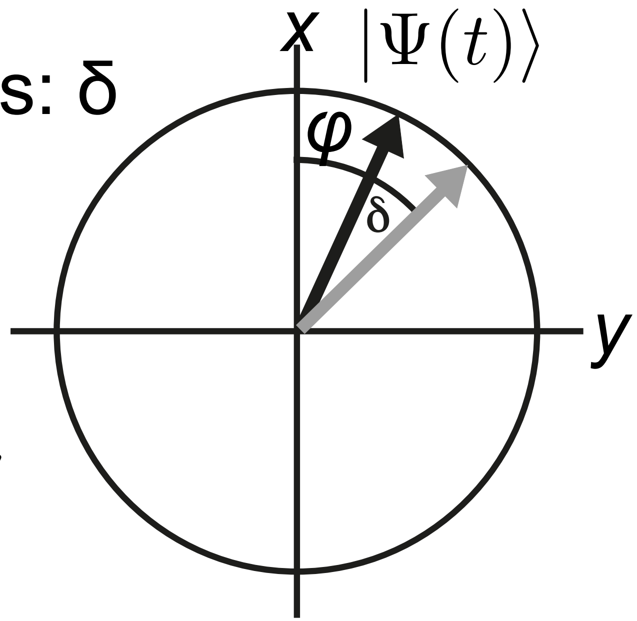
$$|\Psi(t)\rangle = a|0\rangle e^{-i\mathcal{E}_0 t/\hbar} + b|1\rangle e^{-i\mathcal{E}_1 t/\hbar}$$





relative phase: $\varphi = (\mathcal{E}_1 - \mathcal{E}_0)t/\hbar$

additional perturbations: δ



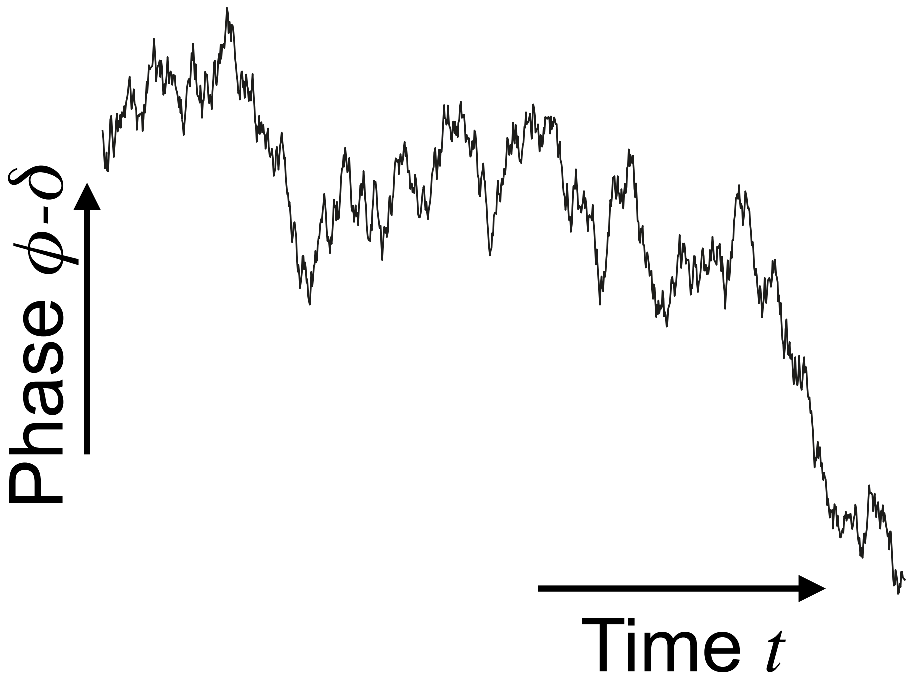
additional precession:

$$\delta(t) = \frac{1}{\hbar}\int_0^t (\delta_{\mathcal{E}_1} - \delta_{\mathcal{E}_0})dt'$$

Random Process

The coupling is in general time dependent

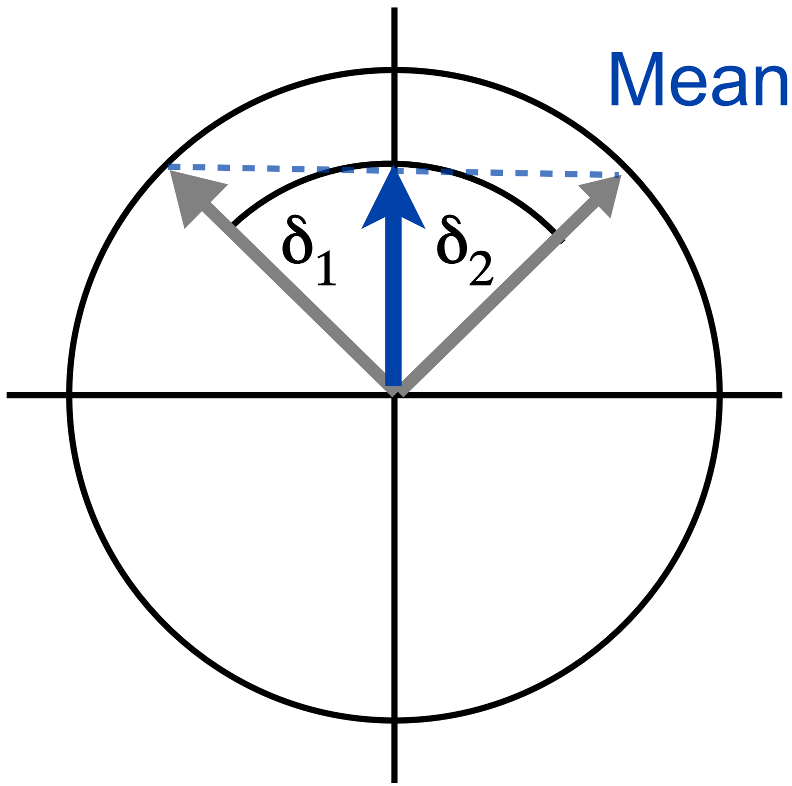
Single qubit : diffusion process



Ensemble Average

In an ensemble, different qubits have different couplings and therefore different precession angles

The average polarization is therefore smaller than that of the individuals



Time Dependence

The coupling is in general time dependent

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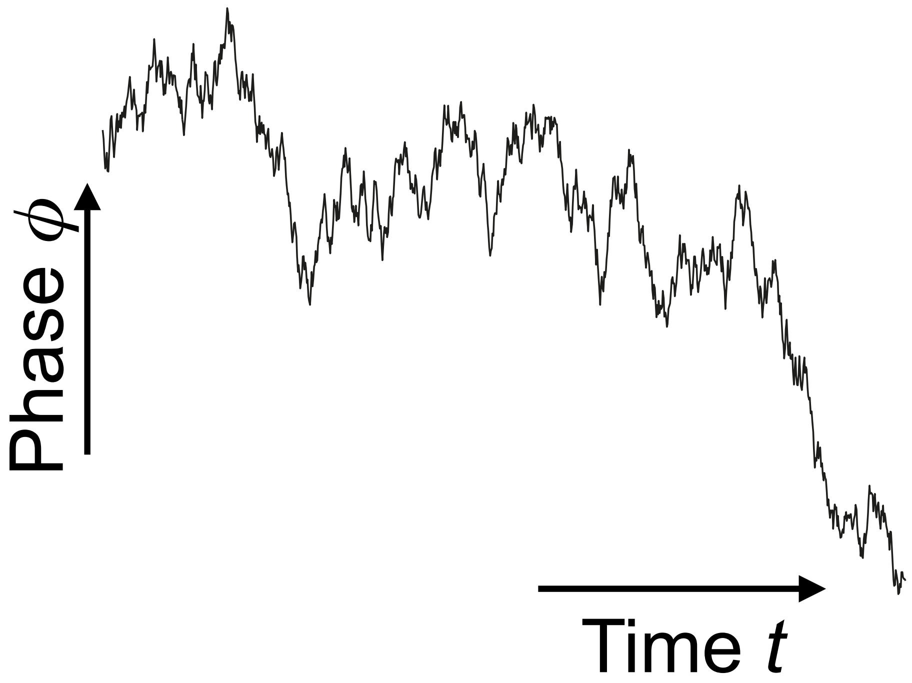
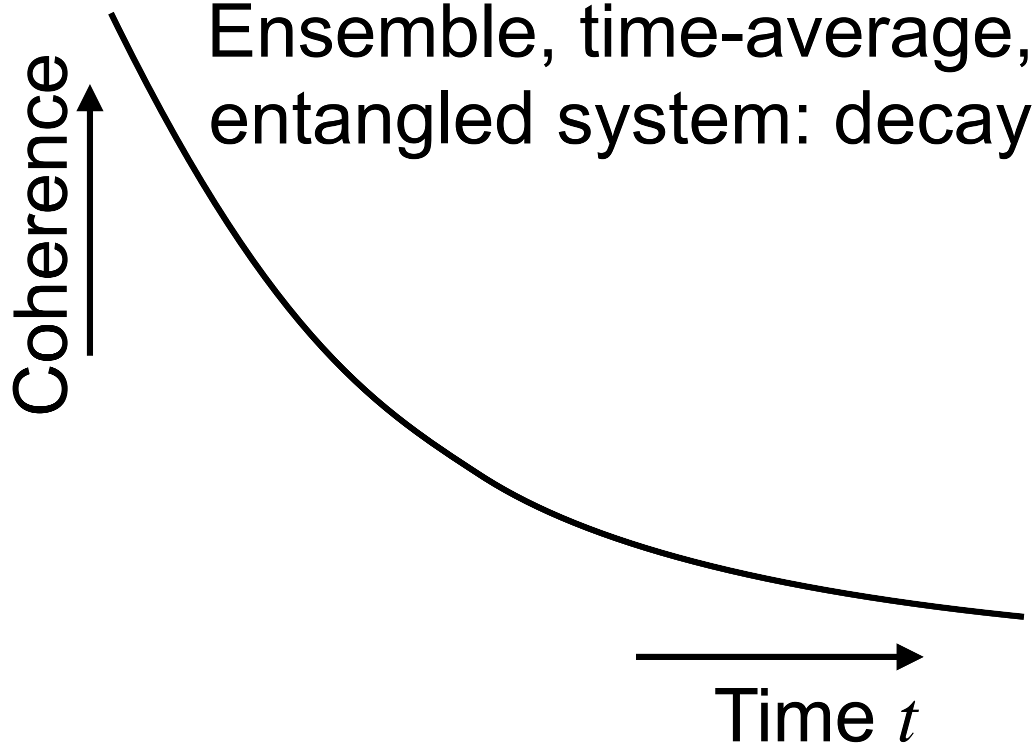


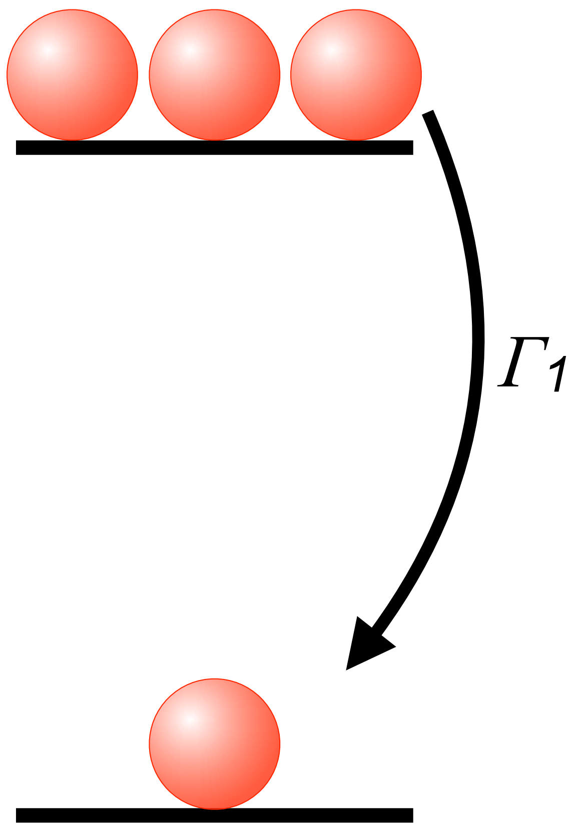
Ensemble, time-average, entangled system: decay

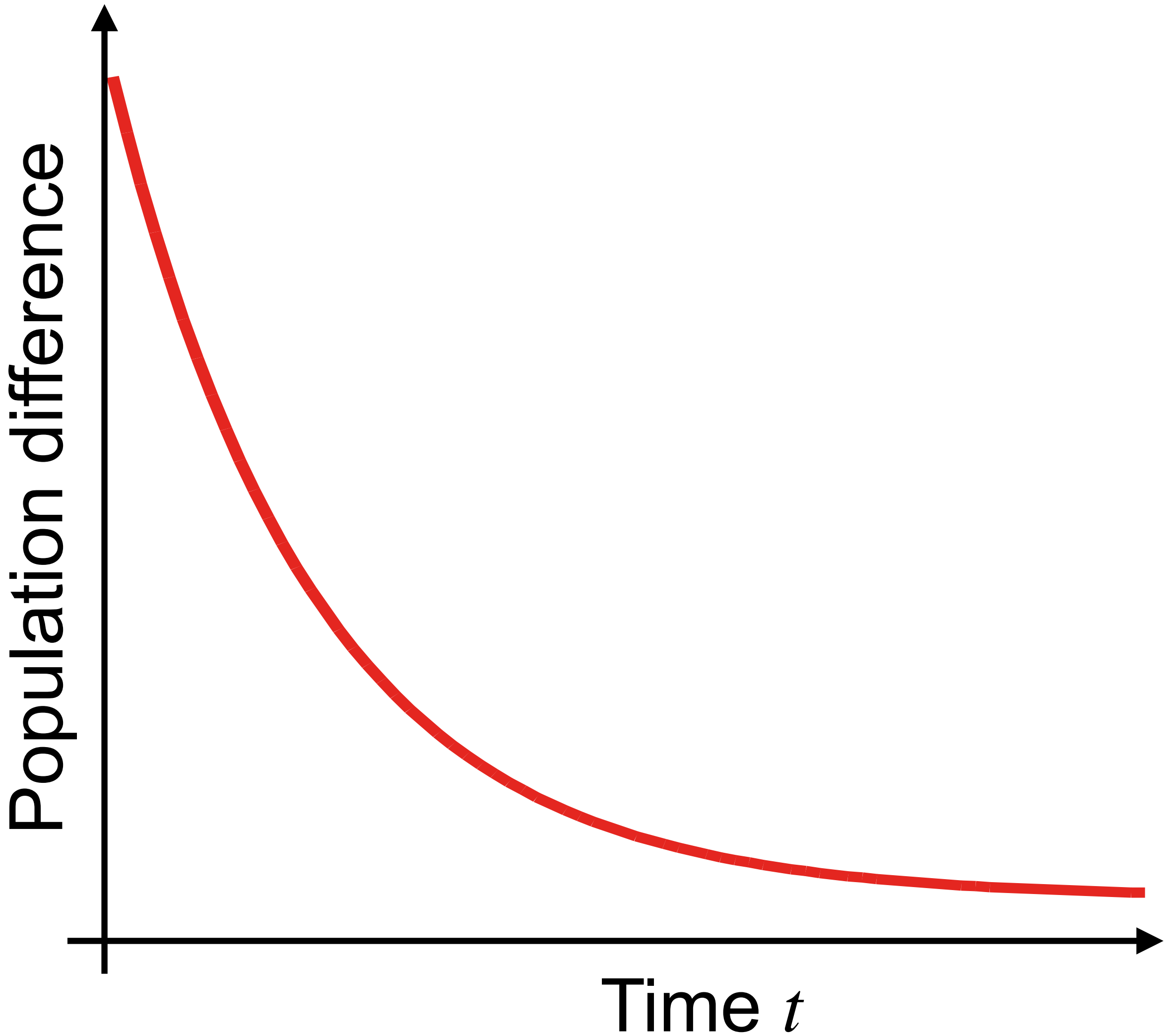


The observed polarization therefore decays

$$\rho_{ij}(t) = \rho_{ij}(0)\, e^{-i(\mathcal{E}_i - \mathcal{E}_j)t/\hbar} e^{-t/T_2}$$

Relaxation of Populations





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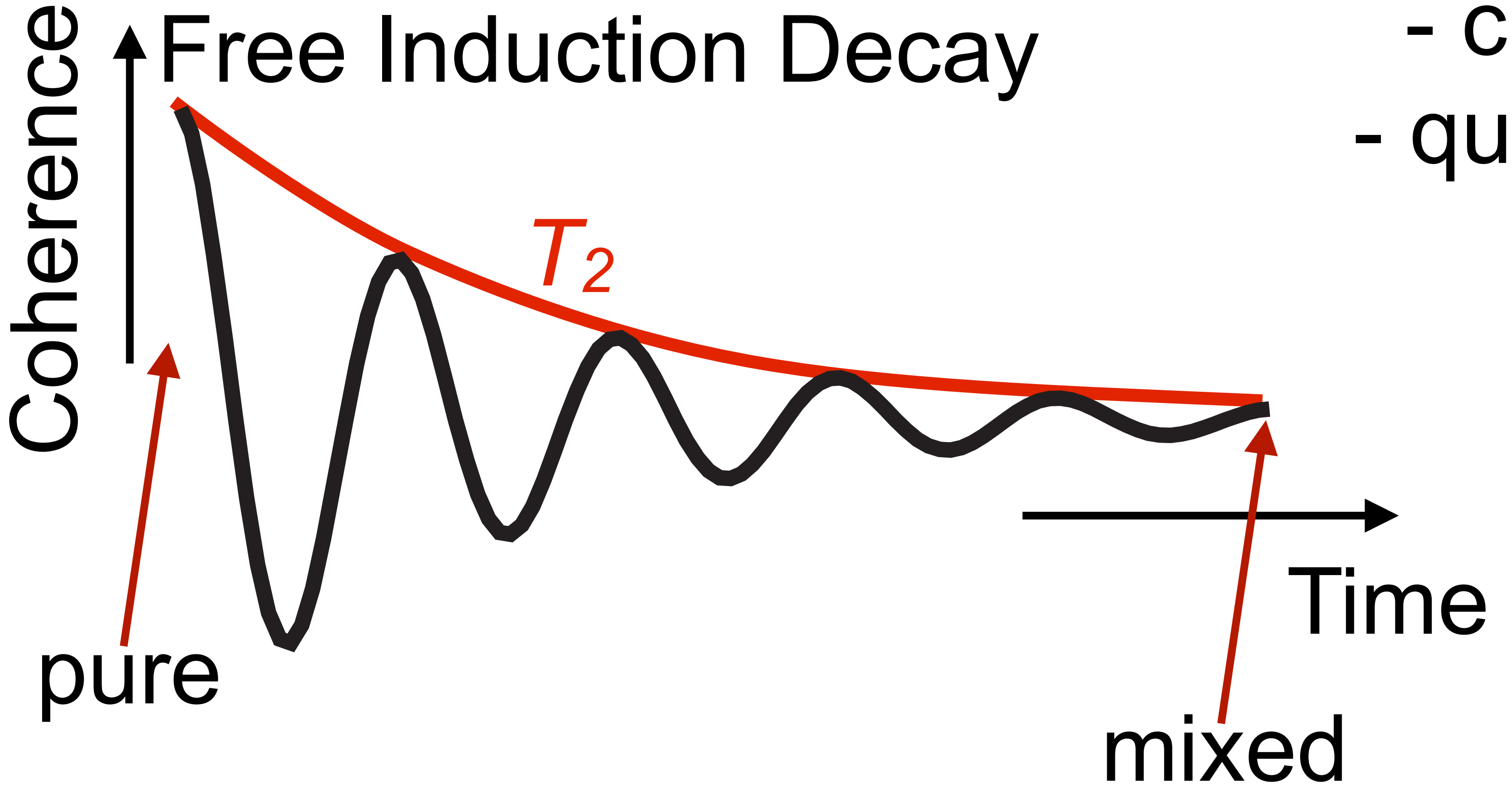

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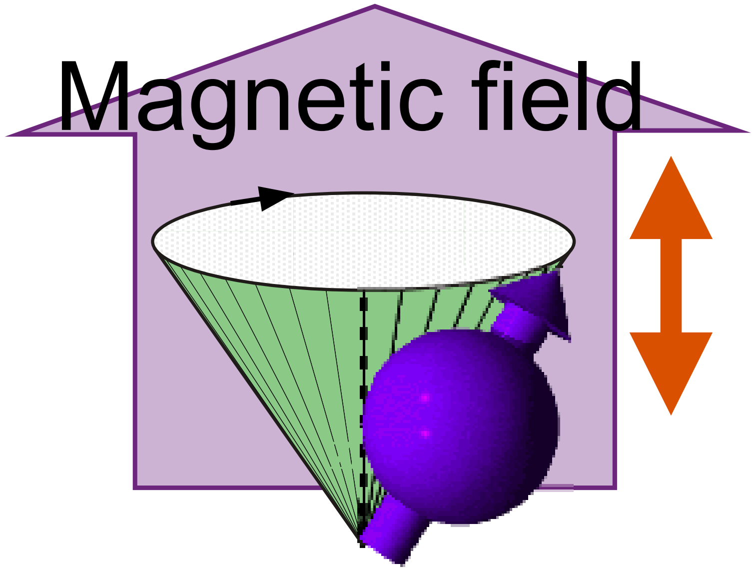

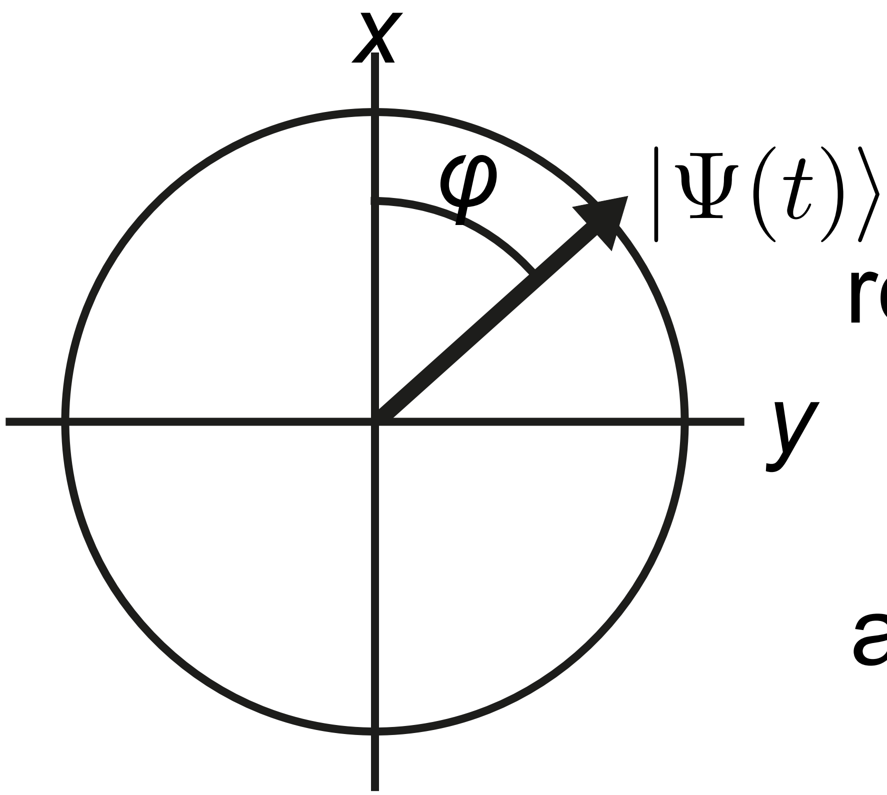

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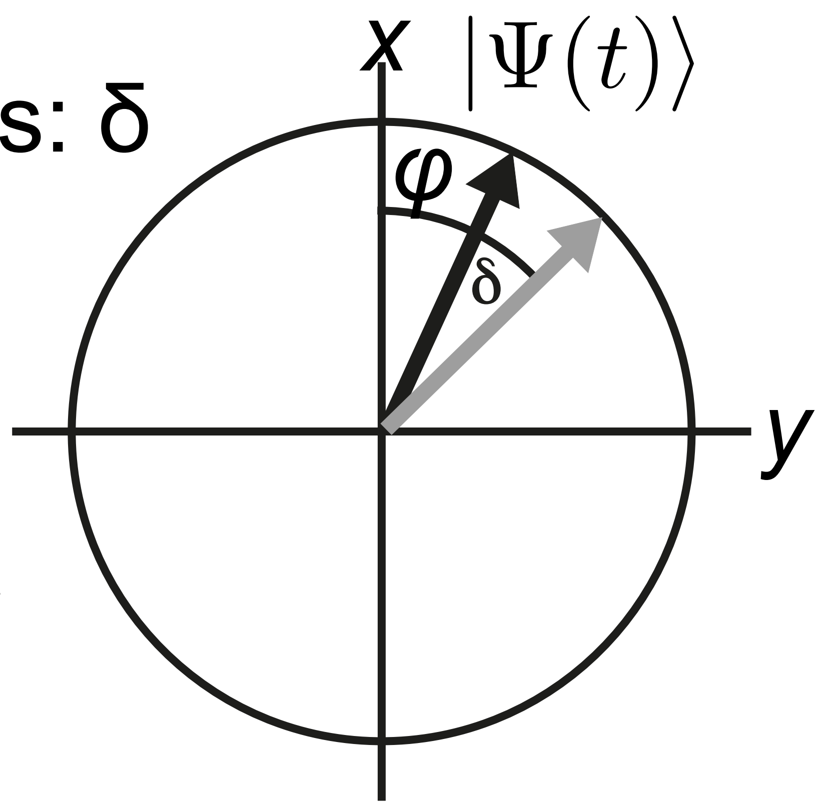

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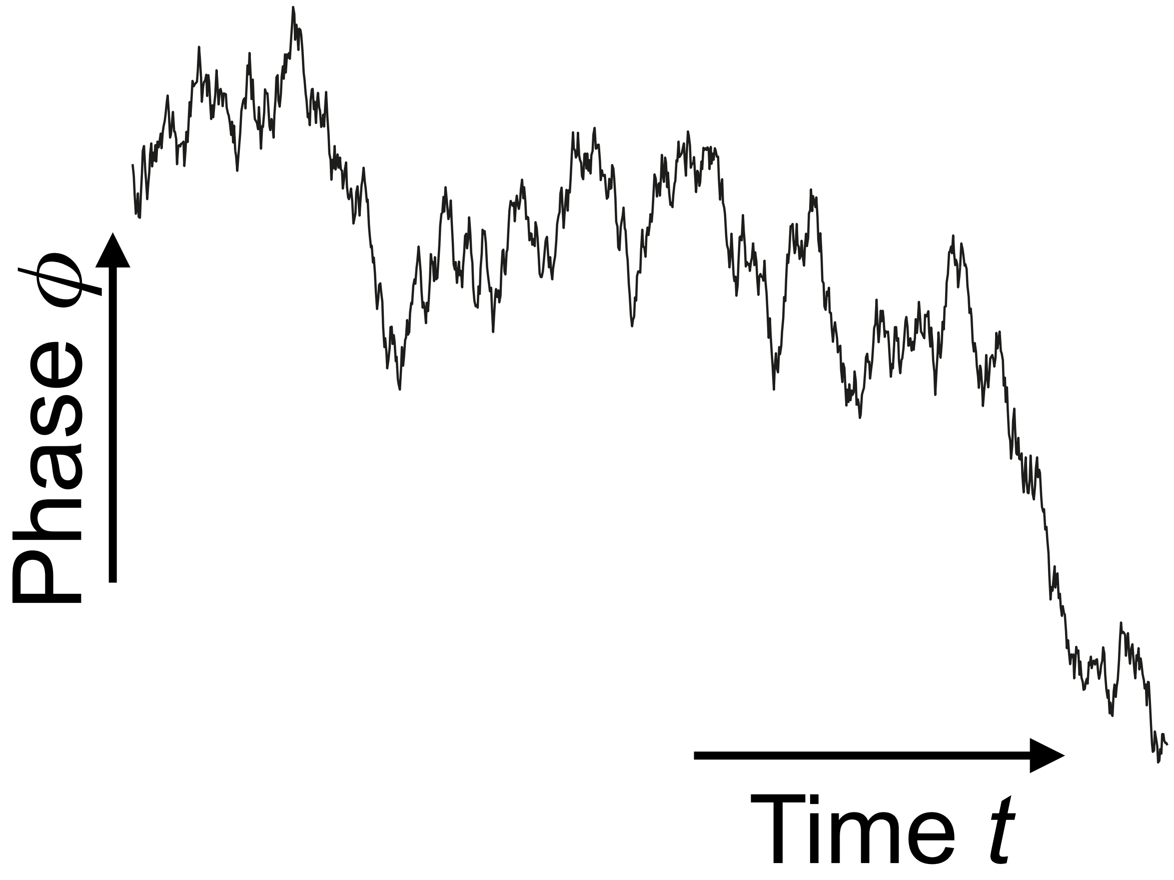

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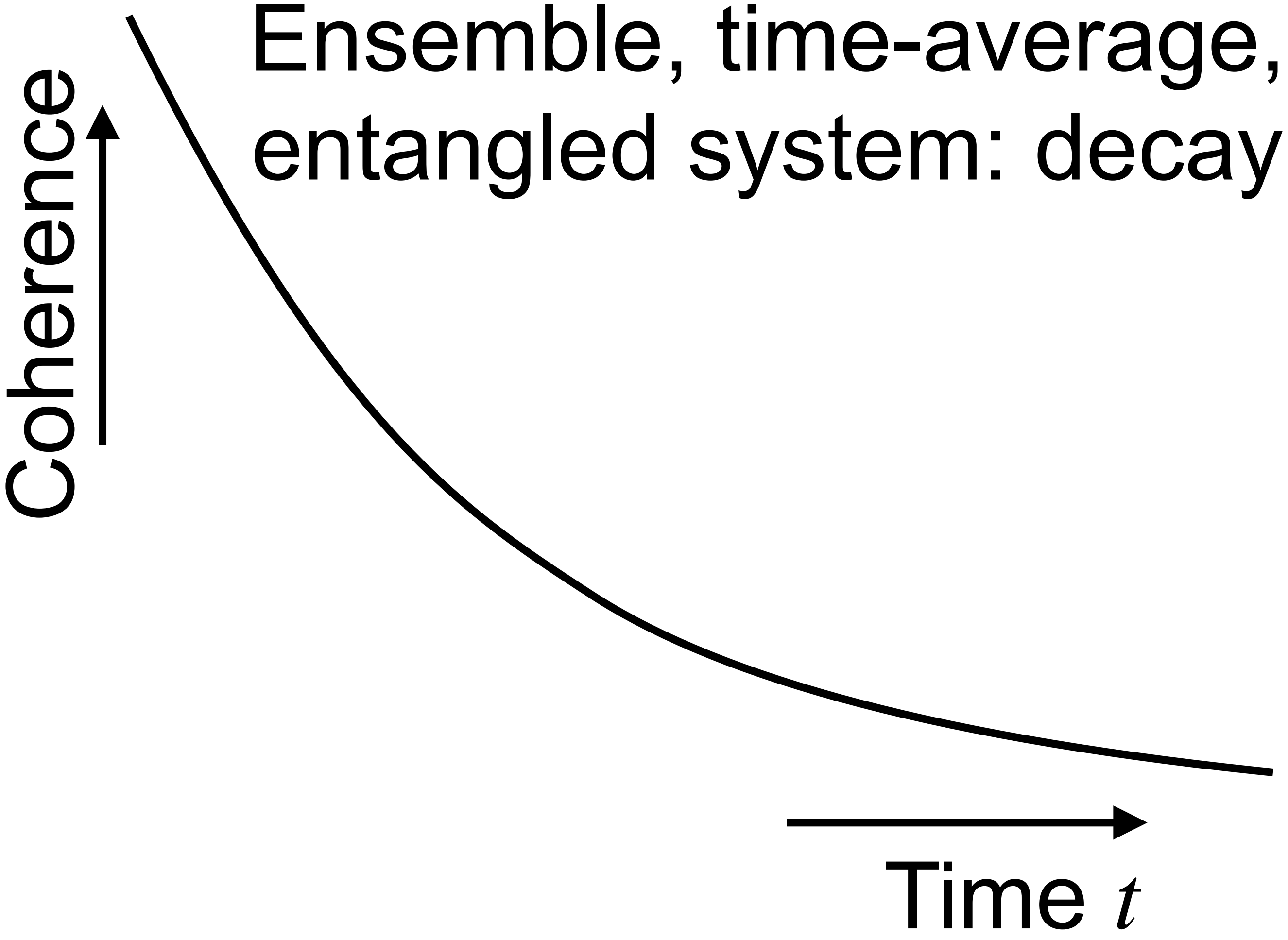

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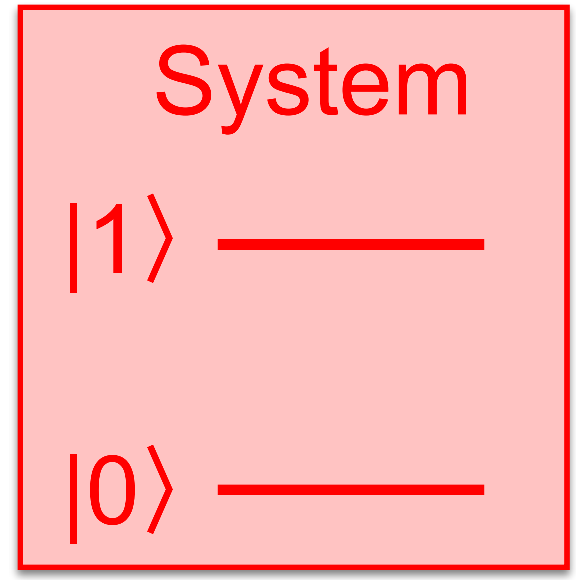
The Spin-Boson Model

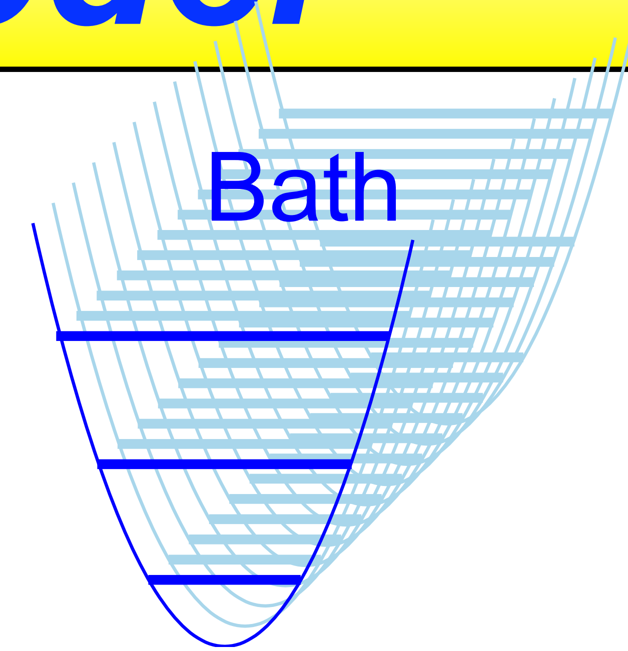
Coupling Hamiltonian
for pure dephasing:

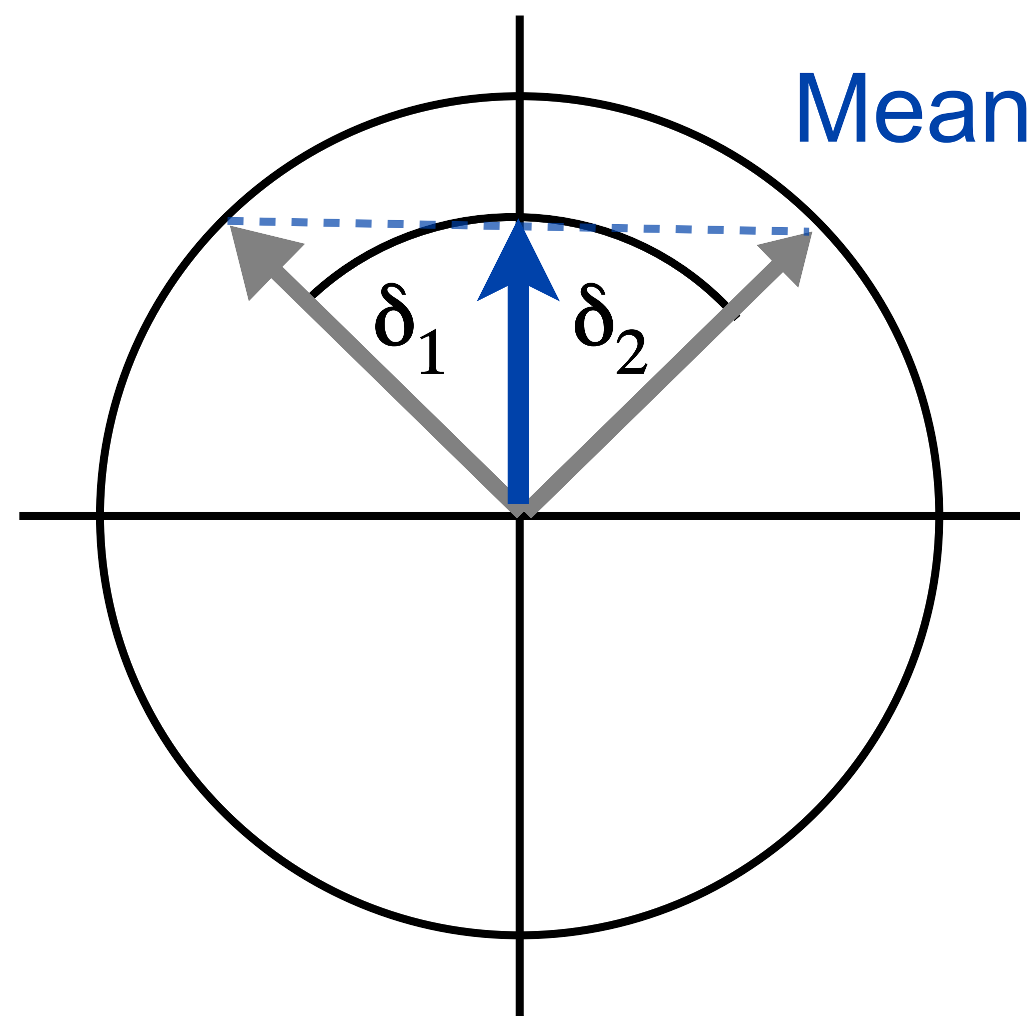
Bath

$$\mathcal{H}_I = \sum_k \sigma_z (g_k b_k^\dagger + g_k^* b_k)$$

System







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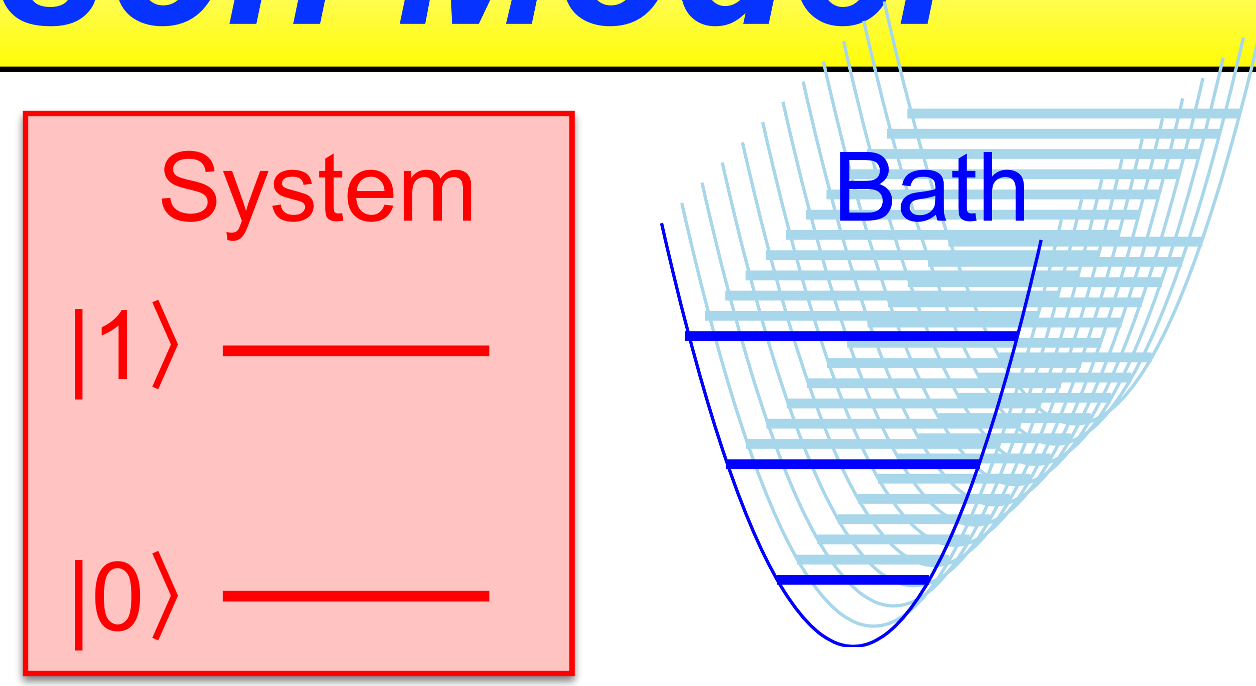
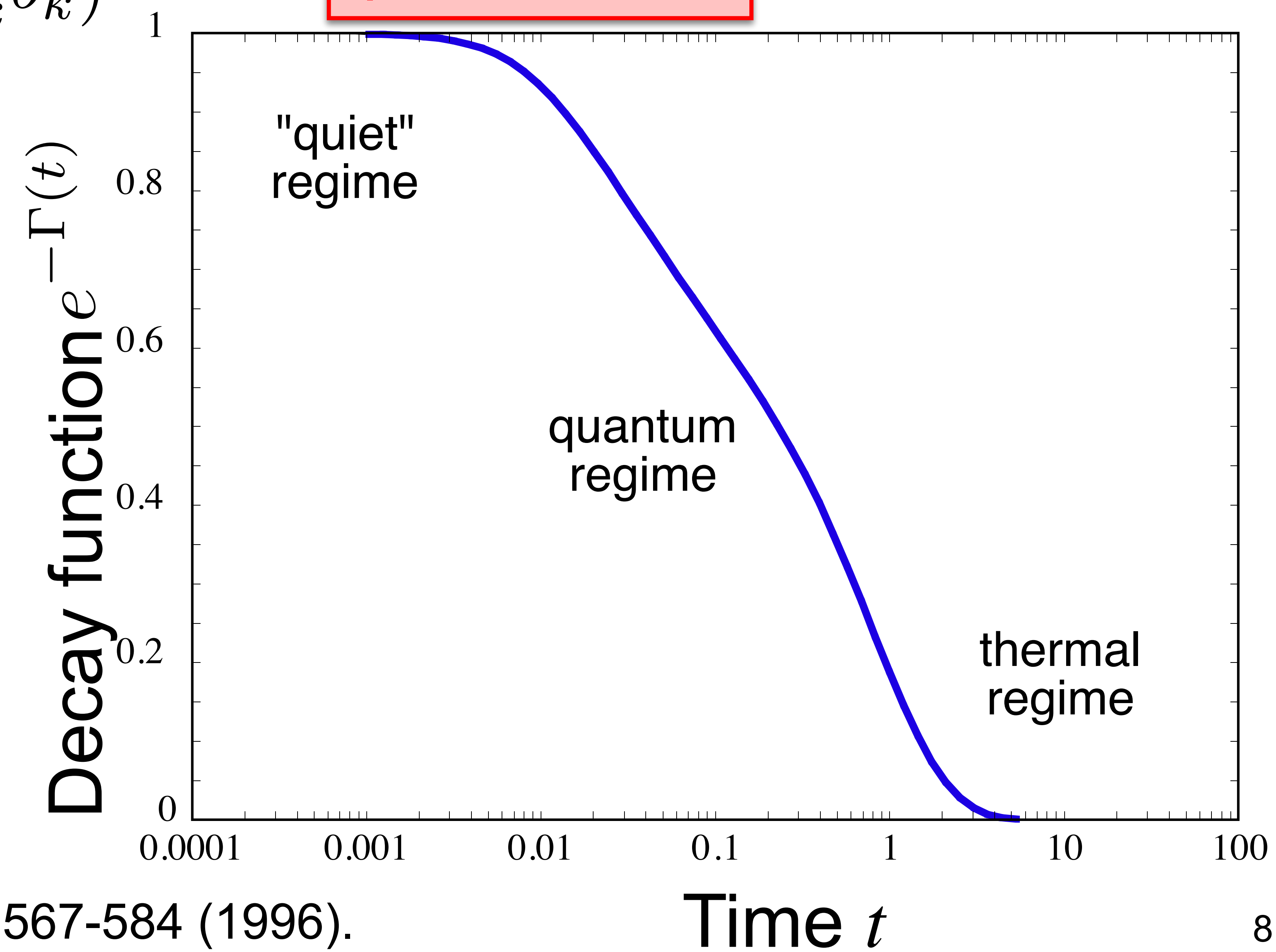




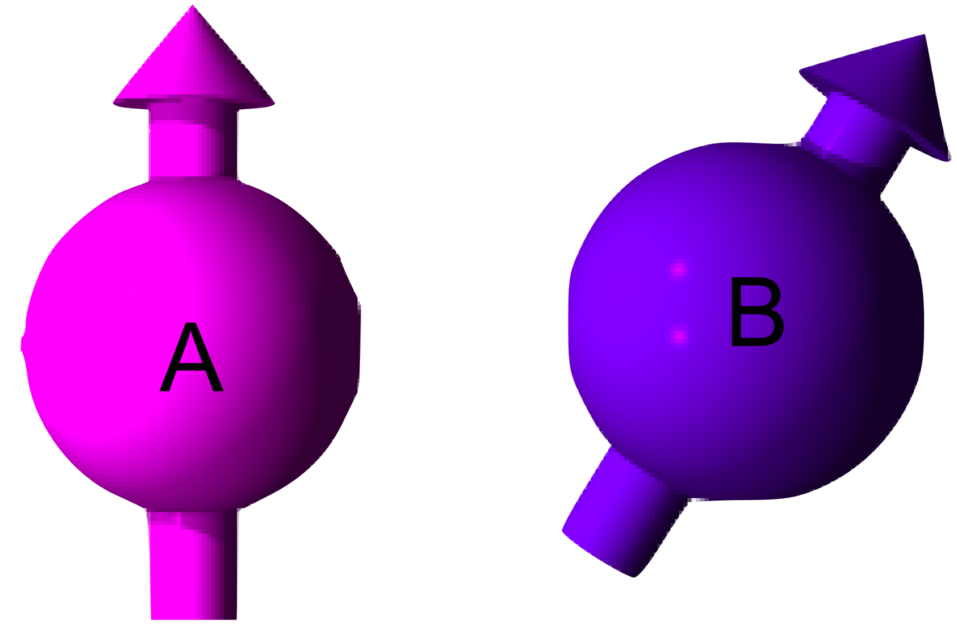
Palma et al., Proc. Roy. Soc. A <u>452</u>, 567-584 (1996).

Spin-Spin Model

Simpler model : 2 spins S = 1/2

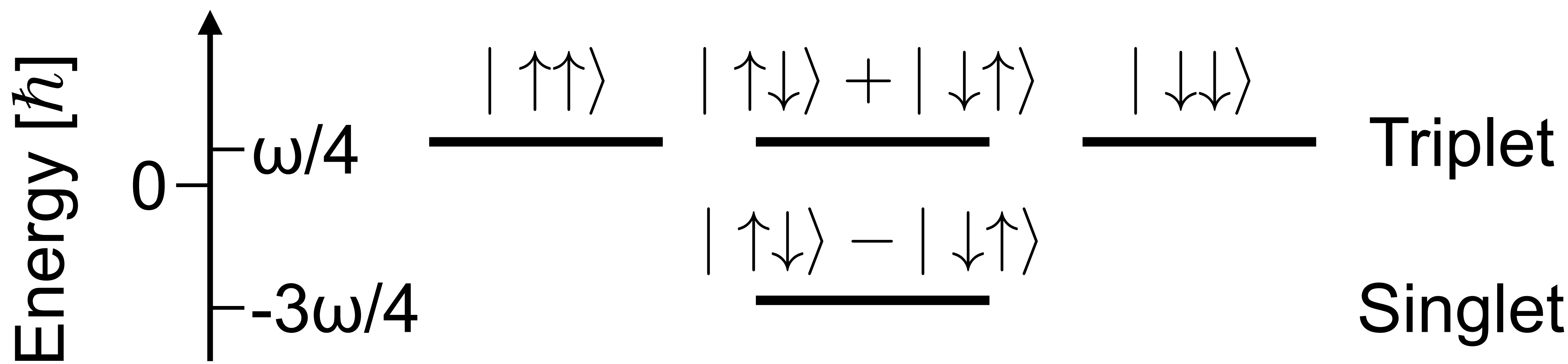
Interaction Hamiltonian:

$$\mathcal{H} = \frac{\omega}{\hbar}\vec{\mathbf{S}}_A \cdot \vec{\mathbf{S}}_B$$



System **Environment**

Eigenstates:



Evolution

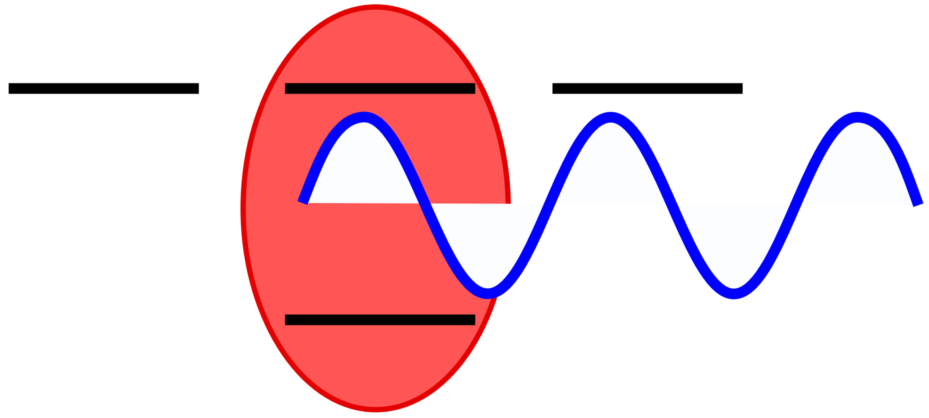
Initial state : general product state

$$|\psi(0)\rangle = \Big(a|\uparrow\rangle + b|\downarrow\rangle\Big)_A \otimes \Big(c|\uparrow\rangle + d|\downarrow\rangle\Big)_B$$

$$= ac|\uparrow\uparrow\rangle + bc|\downarrow\uparrow\rangle + ad|\uparrow\downarrow\rangle + bd|\downarrow\downarrow\rangle.$$

Time evolution

$$\exp\left(\frac{i\omega t}{4}\right)|\psi(t)\rangle = ac|\uparrow\uparrow\rangle + bd|\downarrow\downarrow\rangle$$
$$+\frac{1}{2}[ad(1+e^{i\omega t}) + bc(1-e^{i\omega t})]|\uparrow\downarrow\rangle$$
$$+\frac{1}{2}[ad(1-e^{i\omega t}) + bc(1+e^{i\omega t})]|\downarrow\uparrow\rangle$$



Concurrence: $C = |ad - bc|^2|\sin\omega t|$

Entanglement and Mixing

No entanglement for $|\Psi(0)\rangle = |\uparrow\uparrow\rangle$ or $|\Psi(0)\rangle = |\downarrow\downarrow\rangle$

Maximum entanglement for $|\Psi(0)\rangle = |\uparrow\downarrow\rangle$ or $|\Psi(0)\rangle = |\downarrow\uparrow\rangle$

Time evolution $e^{i\omega t/4}|\psi(t)\rangle = \frac{1}{2}[(1+e^{i\omega t})|\uparrow\downarrow\rangle + (1-e^{i\omega t})|\downarrow\uparrow\rangle]$

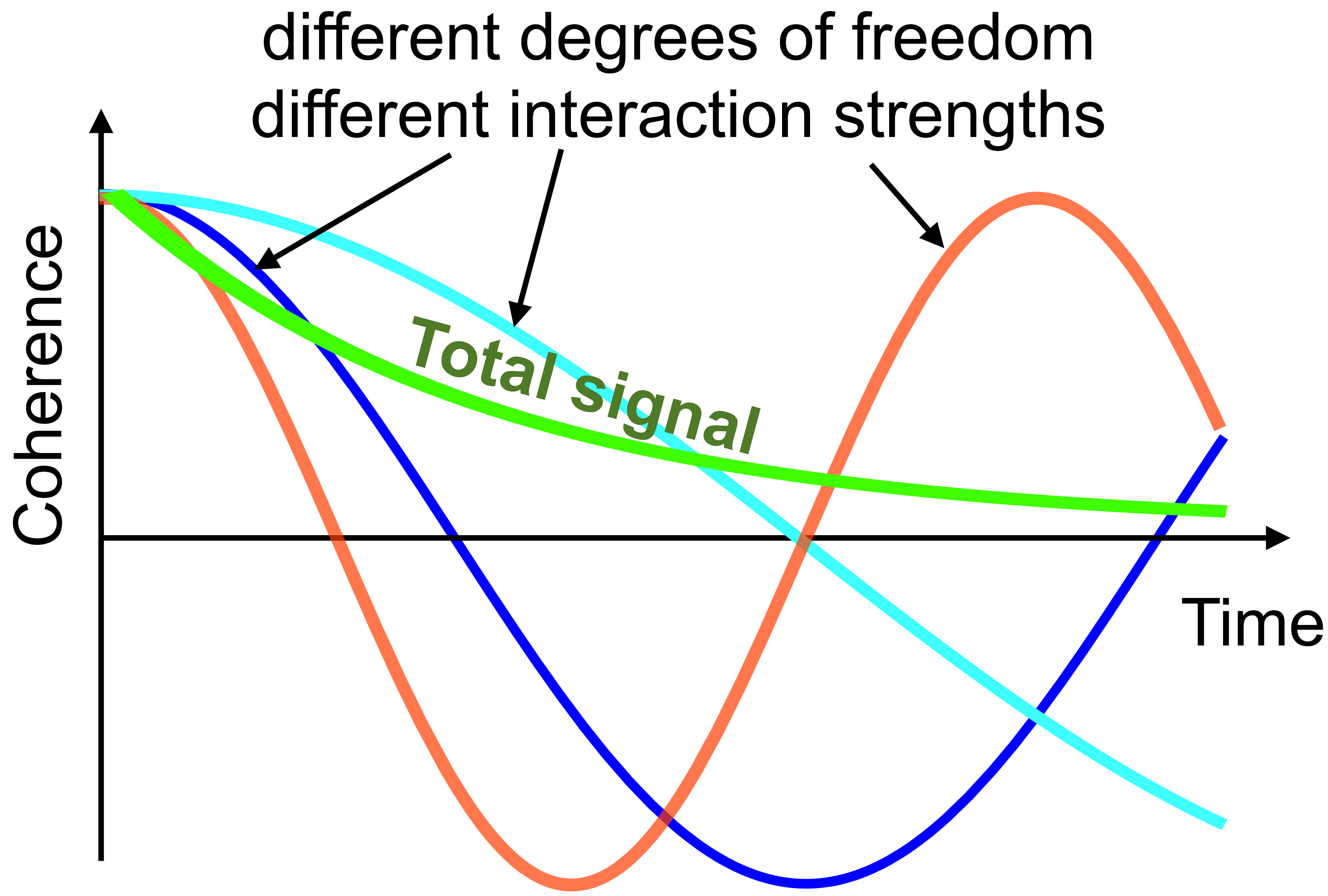
$t = \frac{\pi}{2\omega}$: $\quad |\psi(\frac{\pi}{2\omega})\rangle = e^{-i\pi/8}\frac{1}{2}[(1+i)|\uparrow\downarrow\rangle + (1-i)|\downarrow\uparrow\rangle]$

The corresponding system density operator is

$$\rho_A(\frac{\pi}{2\omega}) = Tr_B|\psi(\frac{\pi}{2\omega})\rangle\langle\psi(\frac{\pi}{2\omega})| = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Several Degrees of Freedom



Example 2

Start in superposition state

$$|\psi(0)\rangle = \frac{1}{2}\left(|\uparrow\rangle + |\downarrow\rangle\right)_A \otimes \left(|\uparrow\rangle - |\downarrow\rangle\right)_B$$

Evolution till $t=\pi/2\omega$

$$|\Psi(\pi/2\omega)> = e^{-i\pi/8}\,\frac{1}{2}\,[|\uparrow>_A \otimes (|\uparrow> - i\,|\downarrow>)_B - i\,|\downarrow>_A \otimes (|\uparrow> + i\,|\downarrow>)_B]$$

$$= e^{-i\pi/8}\,\frac{1}{2}\,[|\uparrow>_A \otimes |\rightarrow>_B - i\,|\downarrow>_A \otimes |\leftarrow>_B]$$

System density operator:

$$\rho_A(\pi/2\omega) = \frac{1}{2}(|\uparrow><\uparrow| + |\downarrow><\downarrow|) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem of Decoherence

Final state: $$\frac{1}{2}\left[|\uparrow\rangle_A \otimes |\rightarrow\rangle_B - i|\downarrow\rangle_A \otimes |\leftarrow\rangle_B\right]$$

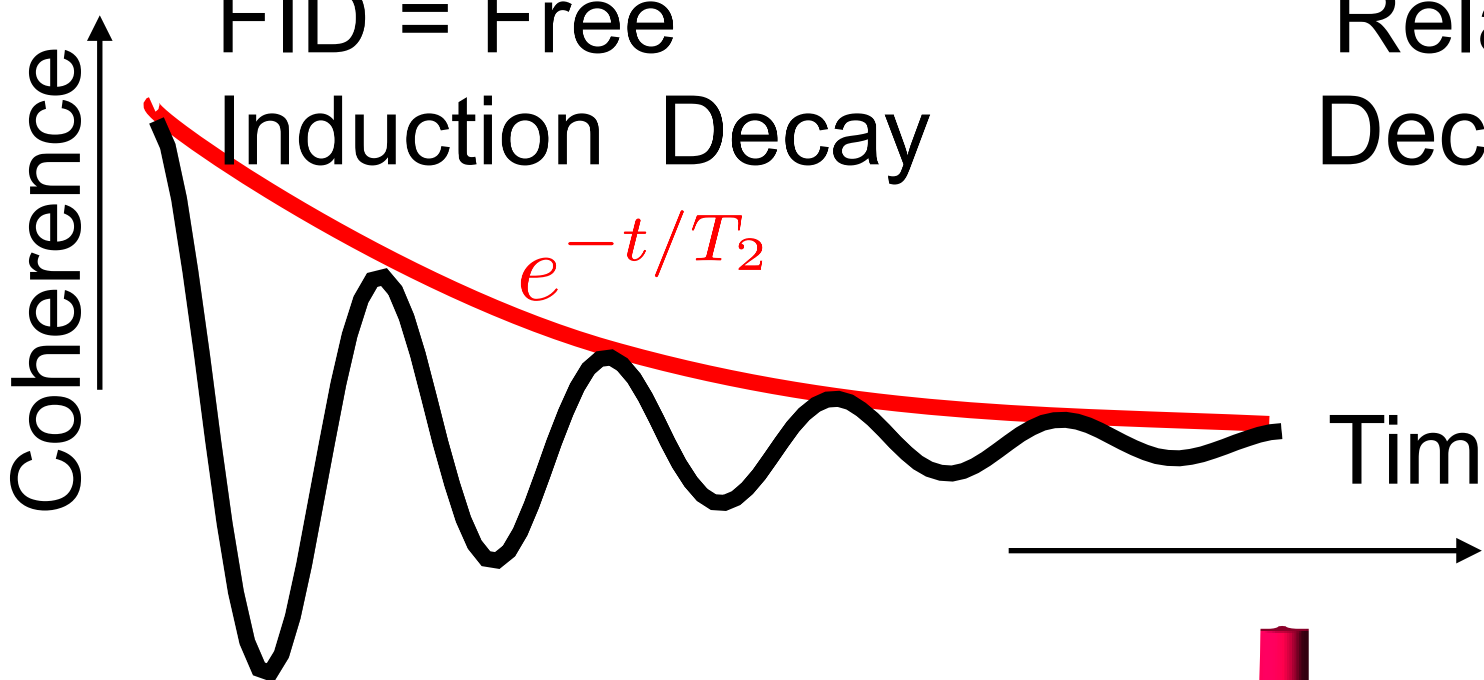
If two mutually orthogonal states of the system of interest become correlated to two mutually orthogonal states of the environment,
all effects of phase coherence between the two system states become lost.

A. J. Leggett, in D. Heiss, editor, Fundamentals of Quantum Information, volume 587 of Lecture Notes in Physics, pages 3–46, Berlin, 2002. Springer Verlag.

The environment has "measured" the system.

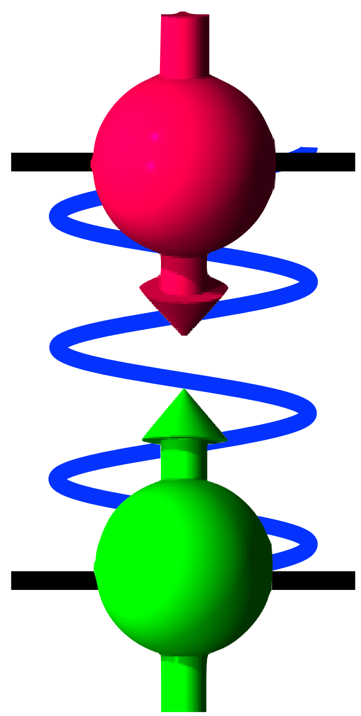
Scaling

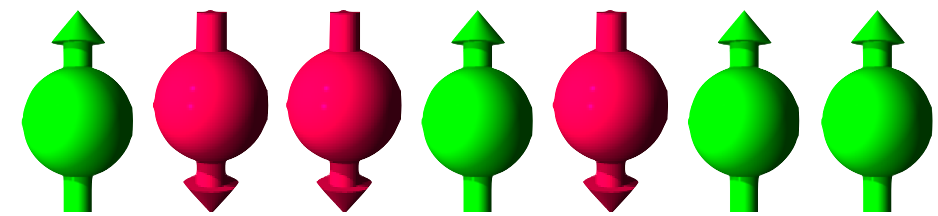
FID = Free Induction Decay



Relaxation = Decoherence

observable magnetization = single qubit coherence

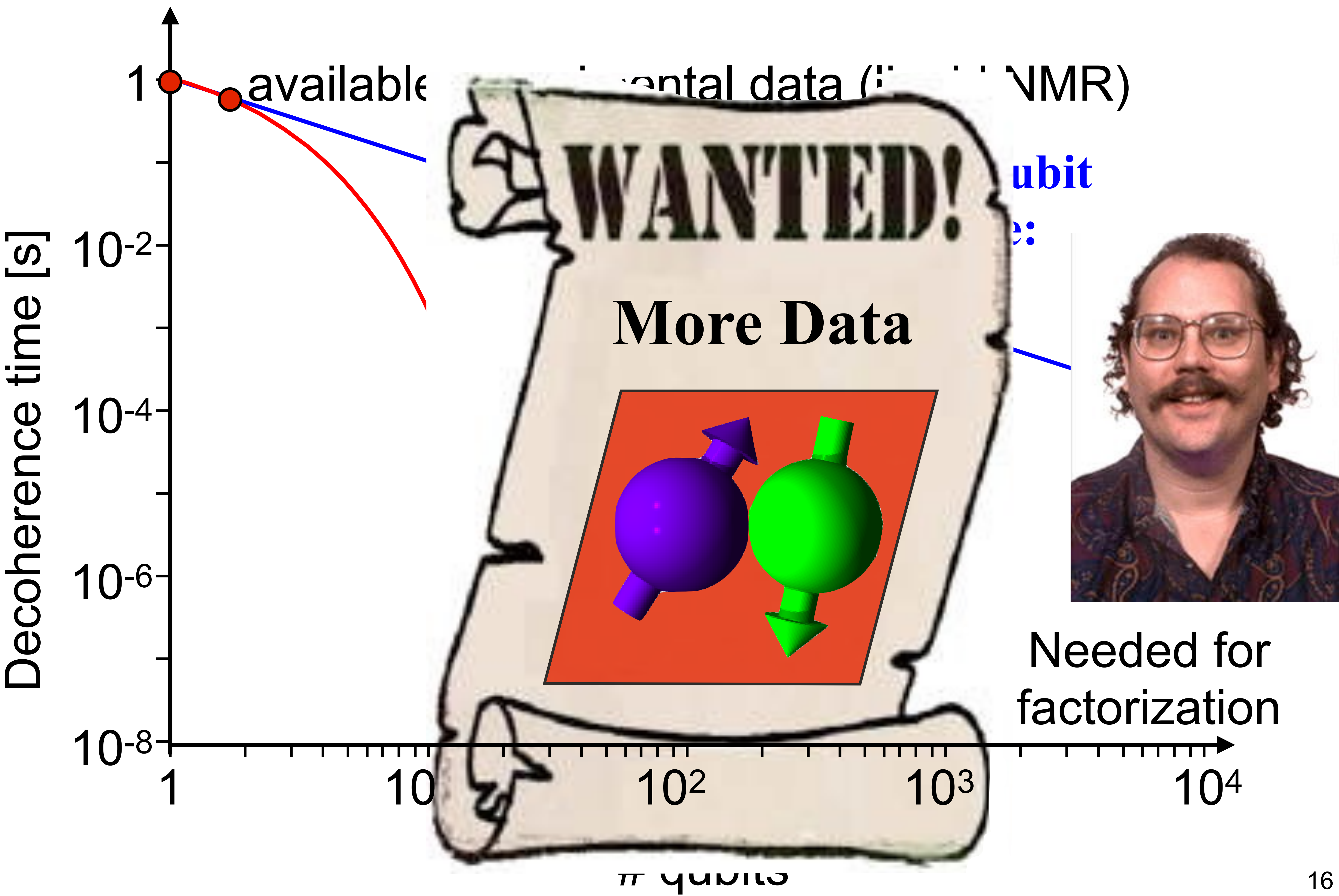




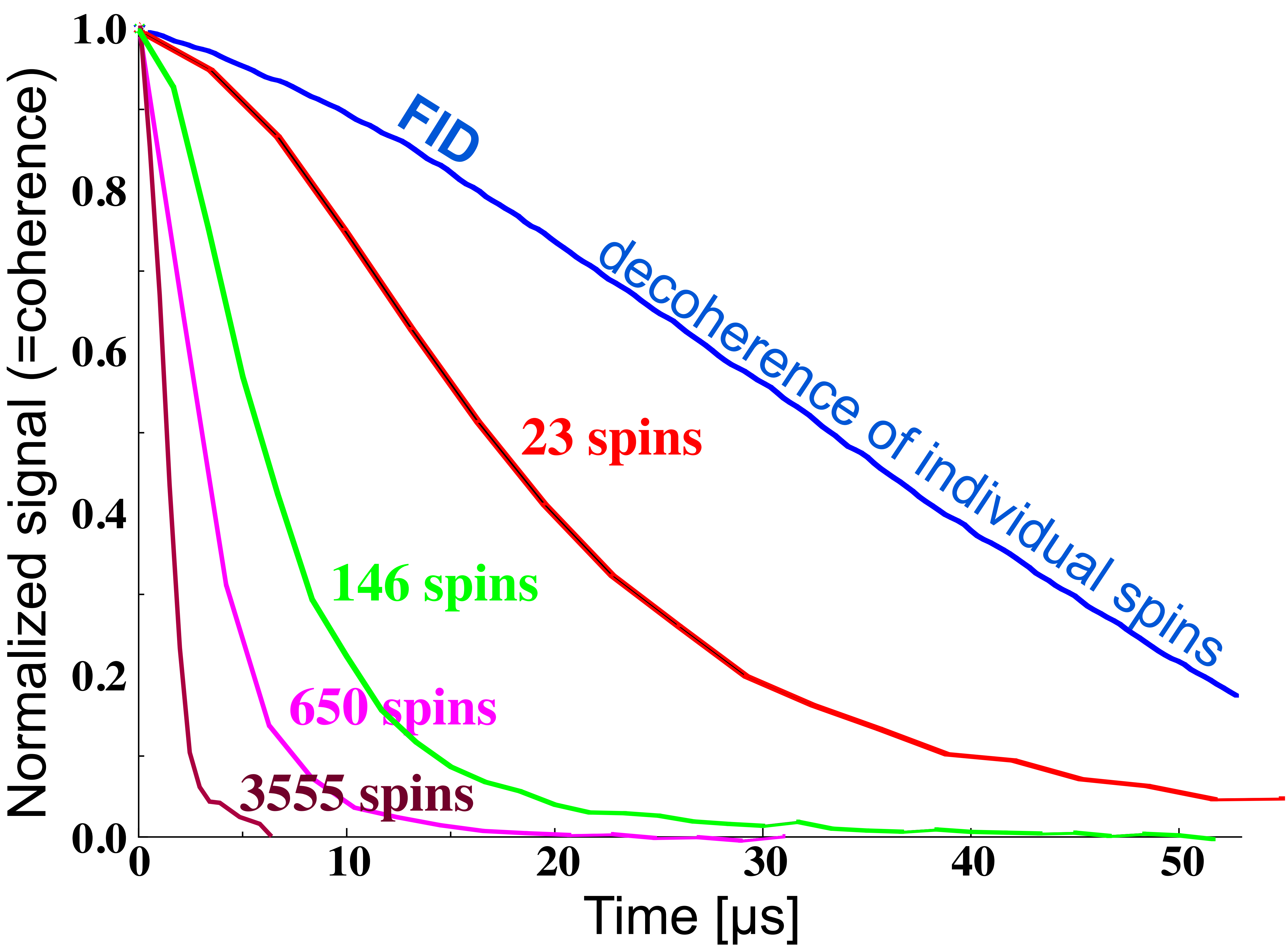
Quantum register involves coherence of many qubits

How fast will a “useful” quantum register lose information ?

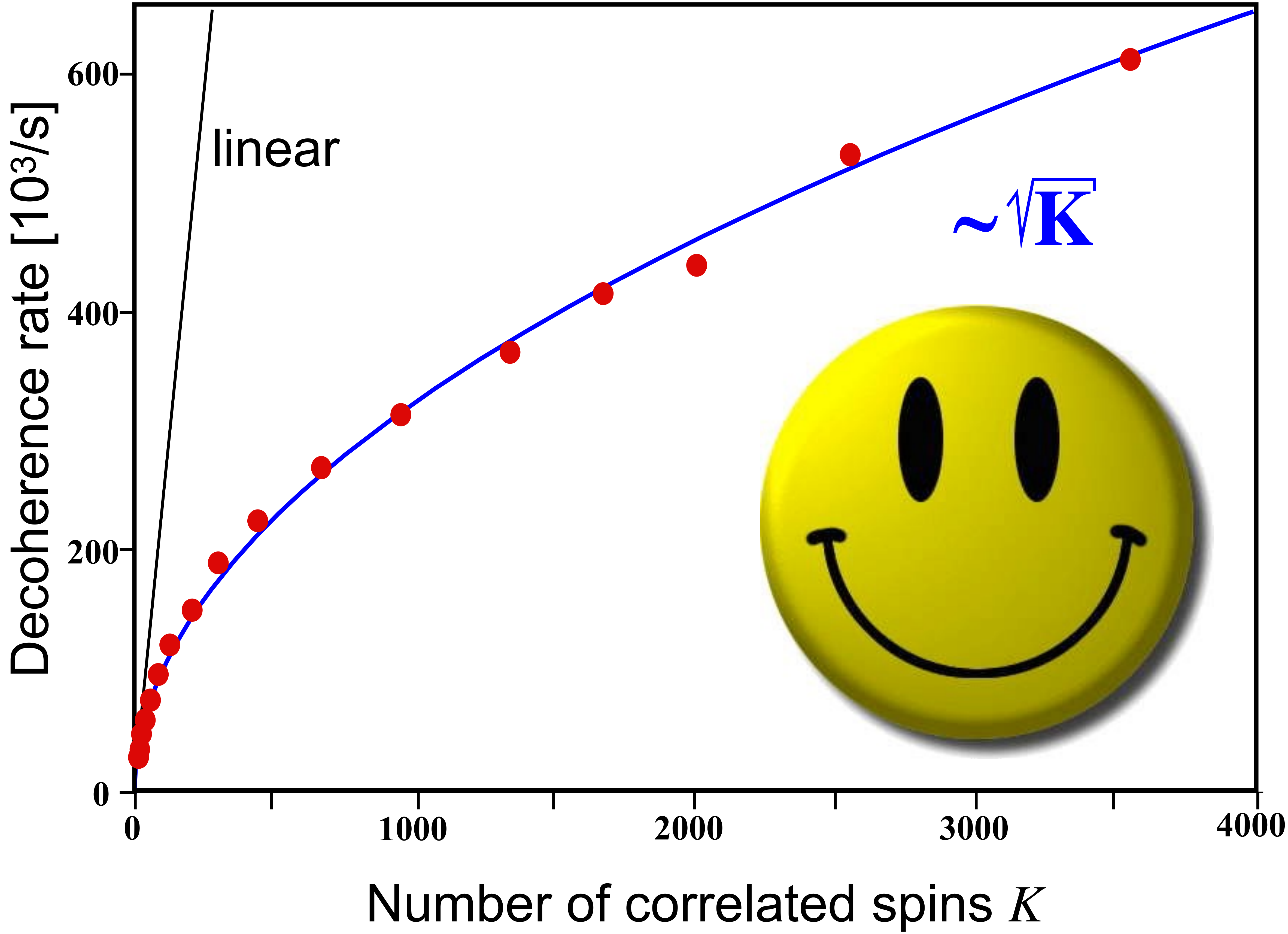
Scaling of Decoherence



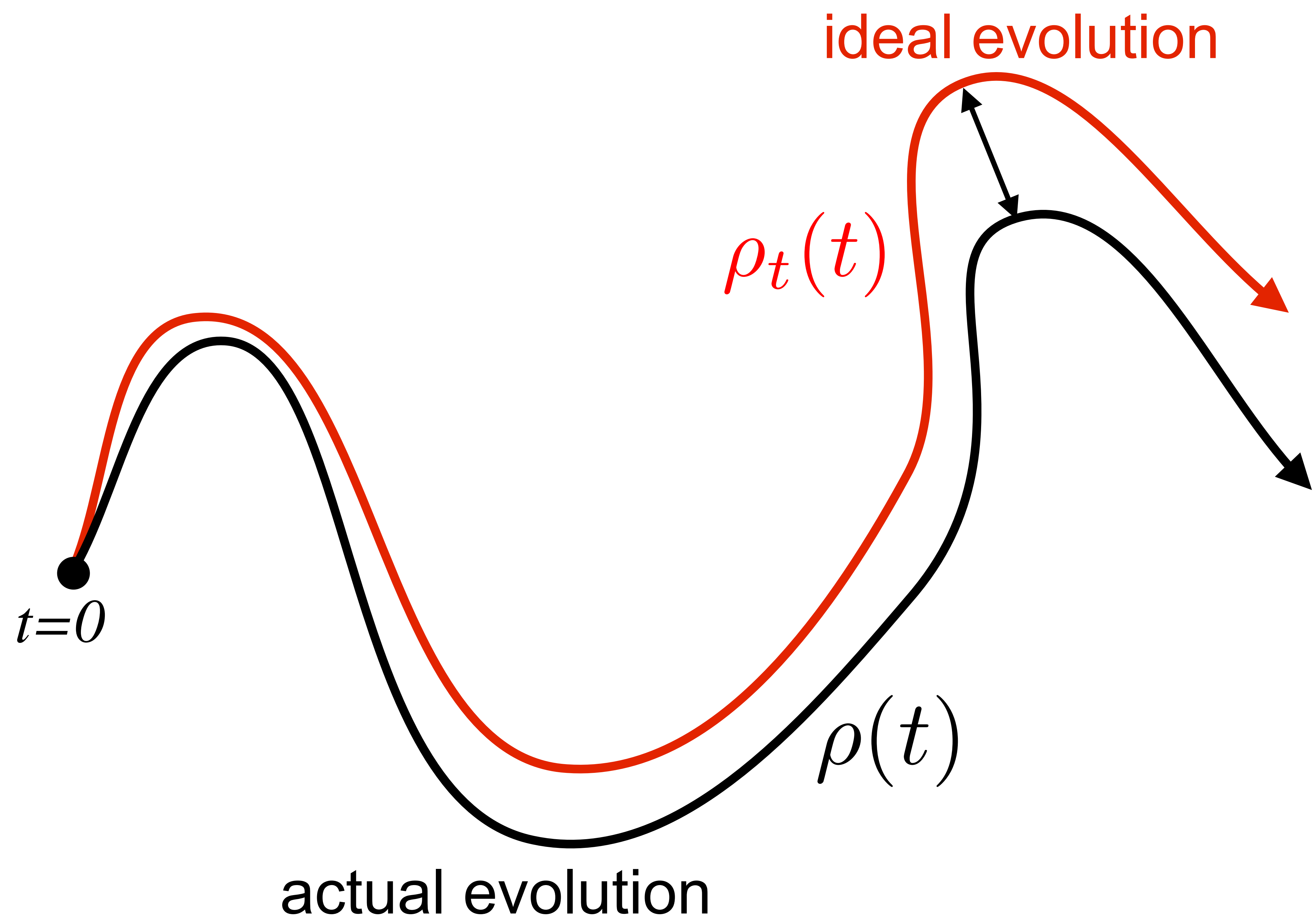
Multiqubit Systems



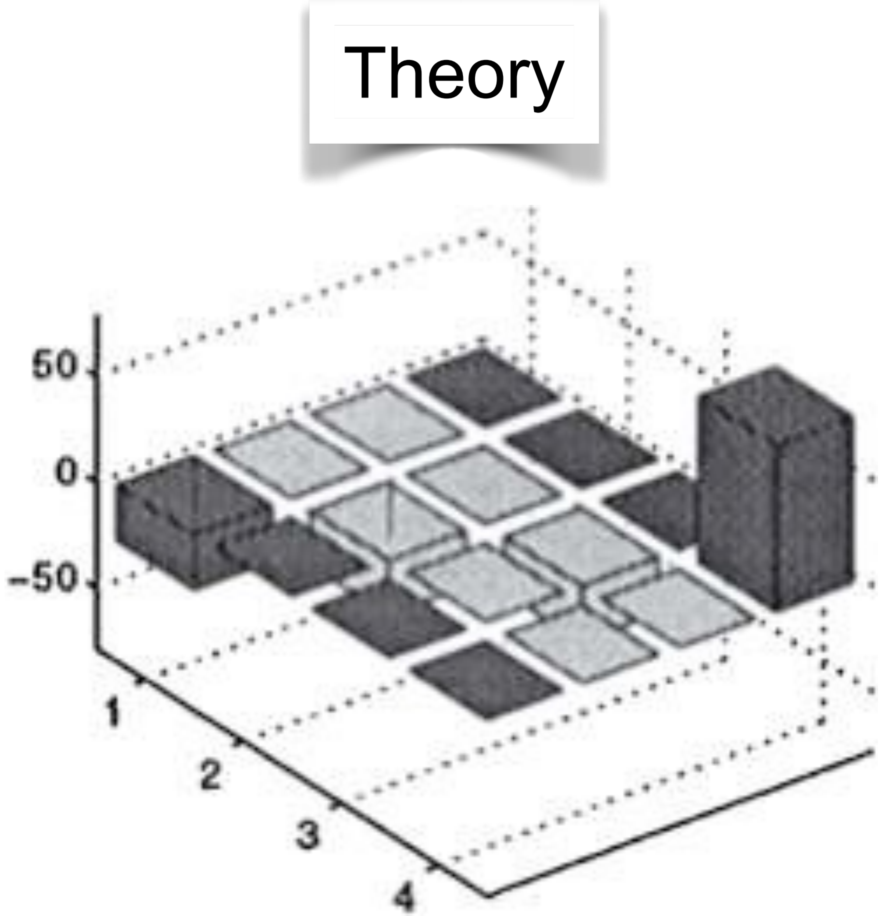
Decoherence Rates

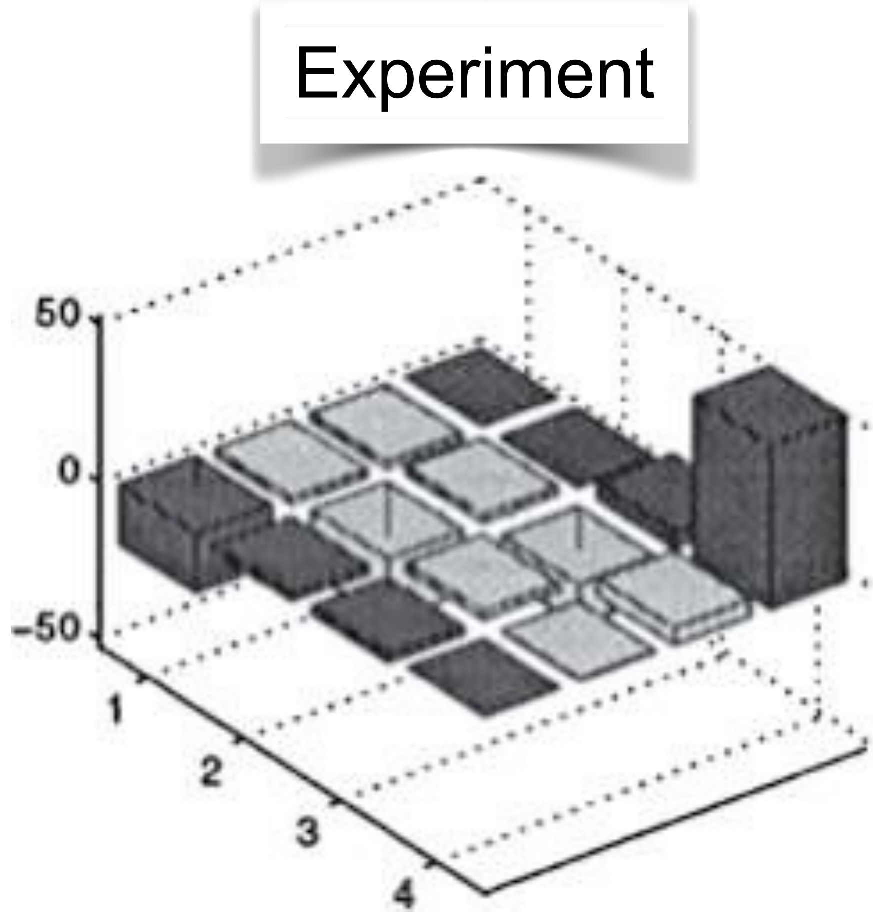


7.3 Quality Measures



Quantum State Tomography

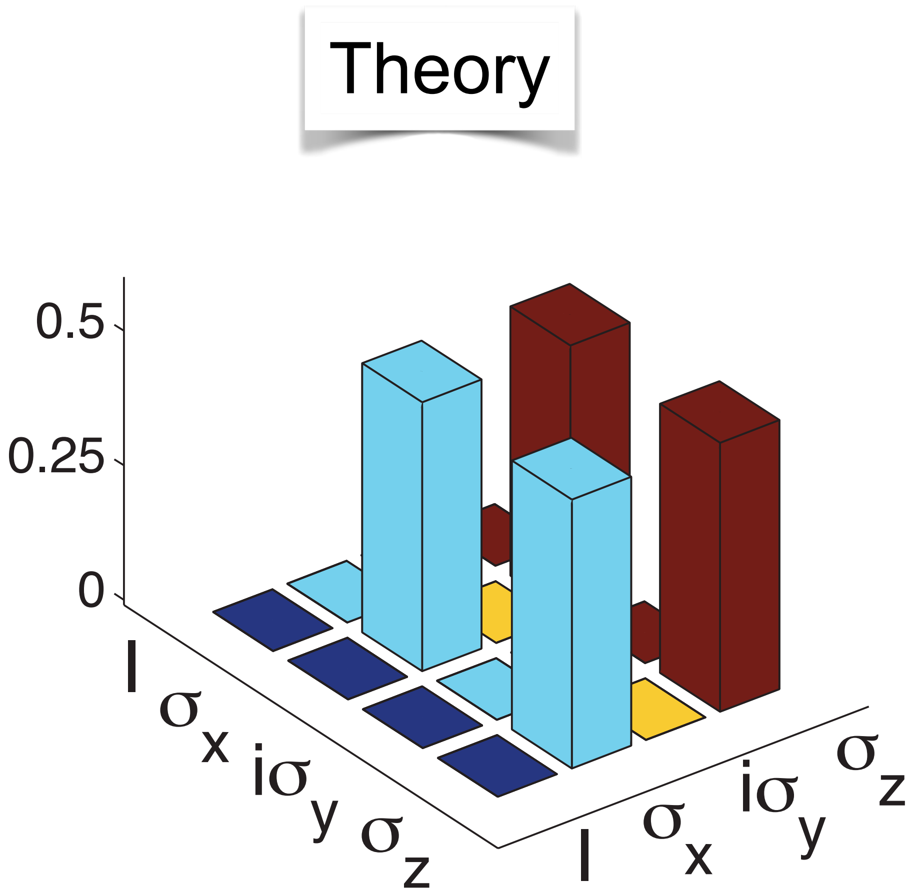


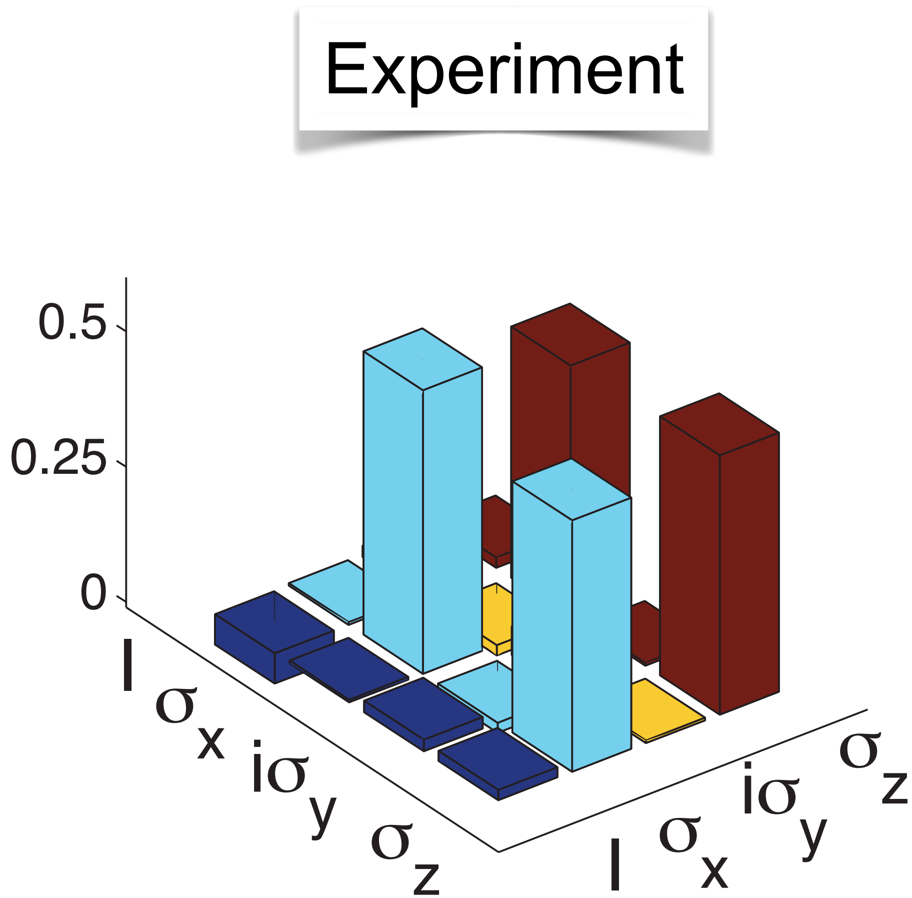


Density operator after a single iteration of the Grover operator

Chuang, Gershenfeld, Kubinec, Phys. Rev. Lett., <u>*80*</u>*, 3408 (19*

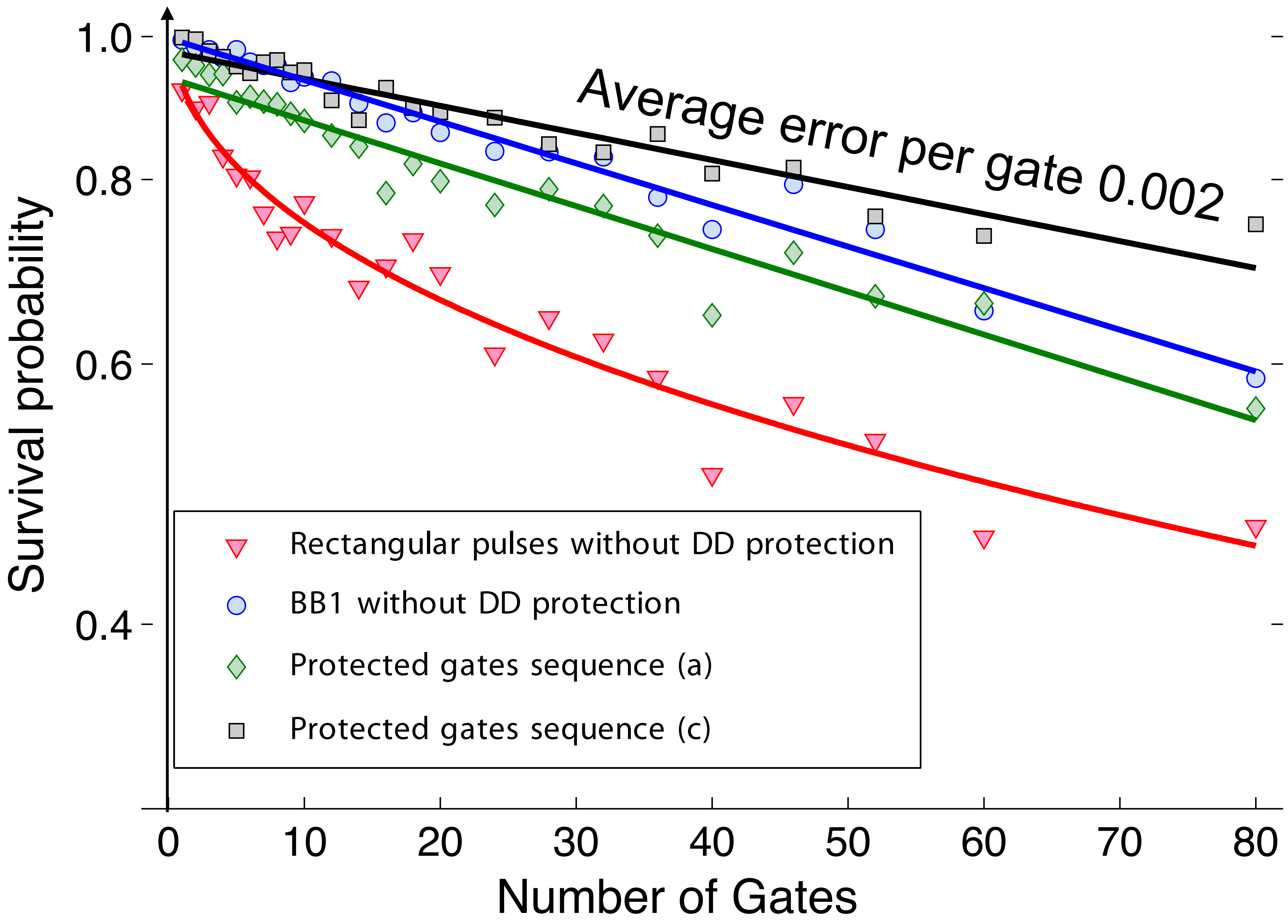
Quantum Process Tomography





Hadamard gate

Quantum Process Tomography



Fighting Errors

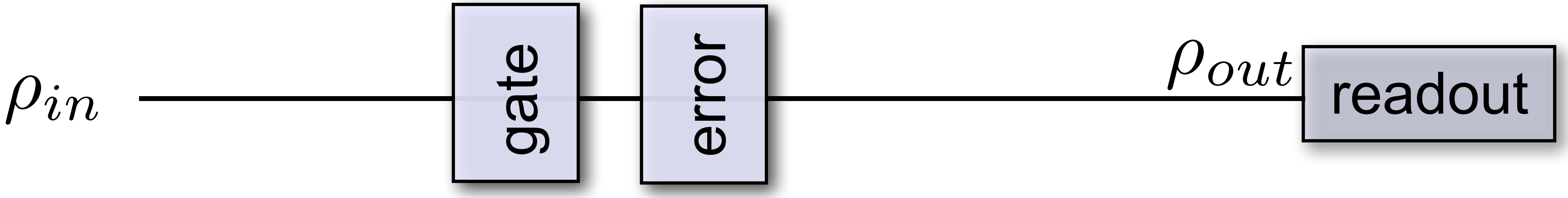
- Optimize the classical apparatus that controls the quantum system to make the gate operations as perfect as possible.
- Design gate operations in such a way that errors in experimental parameters tend to cancel rather than amplify.
- Use error correction schemes.
- Store the information in areas of the Hilbert space that are least affected by the interaction between the system and its environment.
- Use active schemes for decoupling the system from the environment, such as dynamical decoupling.

All schemes must be combined!

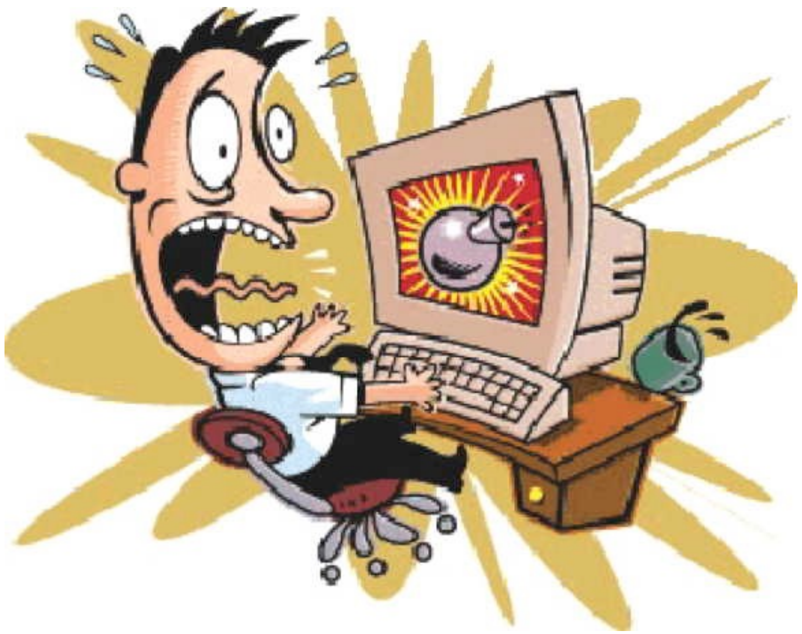
7.4 Quantum Error Correction

7.4 Quantum Error Correction

DD cannot completely eliminate errors

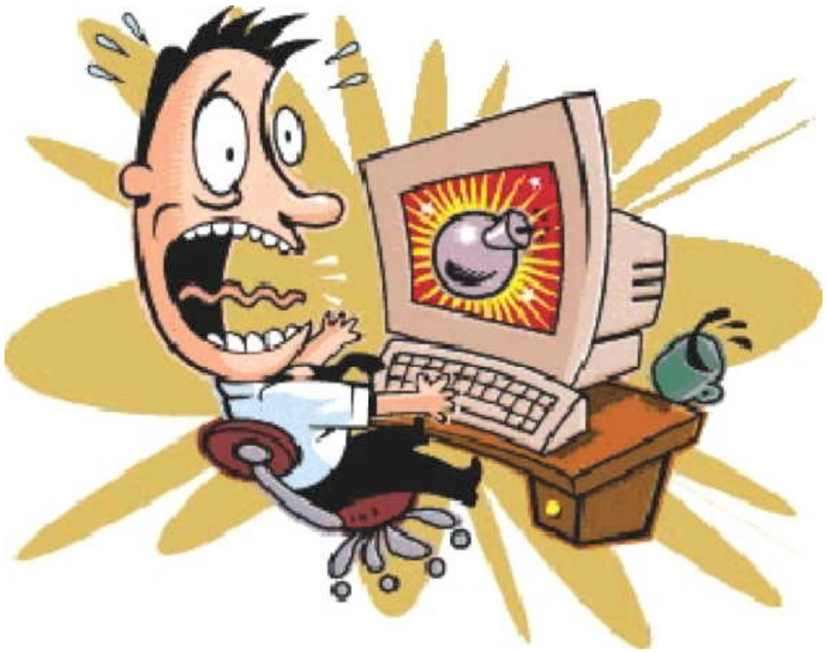


Errors are hard to detect and correct in QIP



Threshold Theorem

Decoherence (=loss of information) leads to errors



... but ...

A quantum computation can be as long as required with any desired accuracy as long as the noise level is below a threshold value

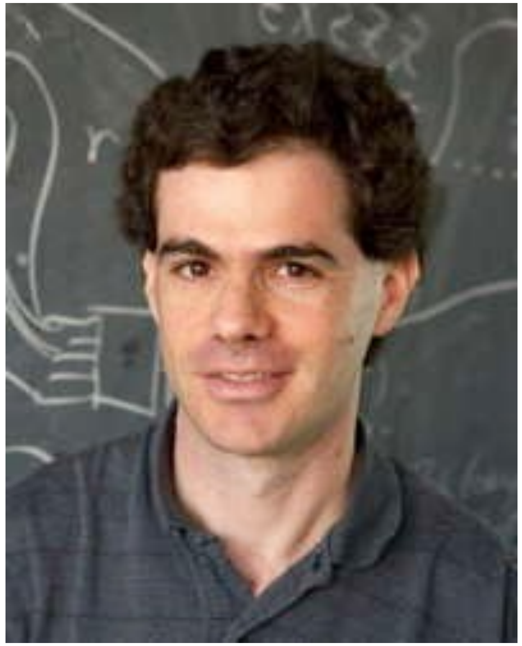
Quantum Error Correction Sonnet

We cannot clone, perforce; instead, we split
Coherence to protect it from that wrong
That would destroy our valued quantum bit
And make our computation take too long.

Correct a flip and phase - that will suffice.
If in our code another error's bred,
We simply measure it, then God plays dice,
Collapsing it to X or Y or Zed.

We start with noisy seven, nine, or five
And end with perfect one. To better spot
Those flaws we must avoid, we first must strive
To find which ones commute and which do not.

With group and eigenstate, we've learned to fix
Your quantum errors with our quantum tricks.



Daniel Gottesman

Classical Digital Information

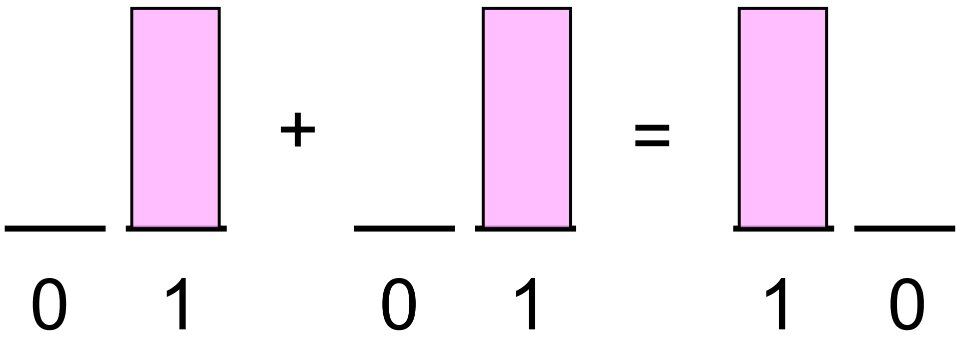
Digital Information is inherently stable

A TTL signal is defined as "low" or L when between 0V and 0.8V with respect to the ground terminal, and "high" or H when between 2V and 5V.

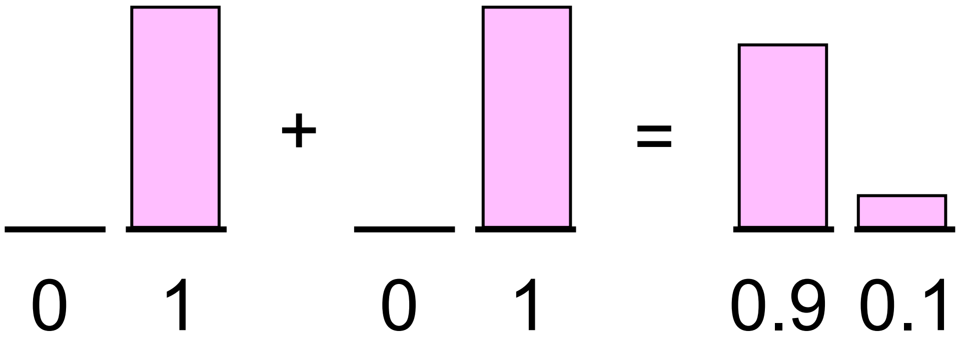
- Small voltage error does not affect information
- Only possible error : bit flip 0 ⇔ 1

Classical Error Correction

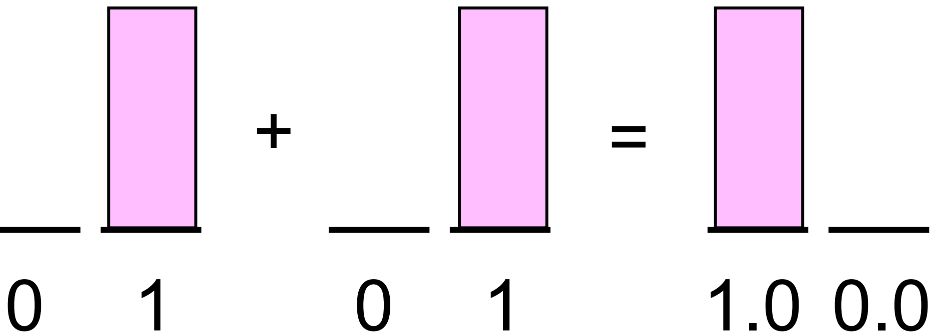
Ideal



Real



Recalibrate



Classical Error Correction

Use redundancy, e.g. $0 \Rightarrow 0_L = 000$ $1 \Rightarrow 1_L = 111$

Single bit error probability $0 < p < 1$

Probability for			
0 error	$(1-p)^3$	$\sim$	$1-3p$
1 error	$3p(1-p)^2$	$\sim$	$3p$
2 errors	$3p^2(1-p)$	$\sim$	$3p^2$
3 errors	p^3	$=$	p^3

After transmission / calculation:

check if all bits identical; if not : flip the one that differs

$001, 010, 100 \rightarrow 000$ $110, 101, 011 \rightarrow 111$

$\Rightarrow$ remaining error probability $\sim 3p^2$

Quantum Error Correction

Ideal:

$$\Psi_{in} \xrightarrow{U} \Psi_{out}$$

$t=0$ $\qquad$ $t=\tau$

Main difficulties:

- **# possible states increases exponentially**
- **Cannot measure qubits during computation**
- **Must maintain phase coherence**

Quantum vs. Classical

Quantum information "more valuable"
but more fragile

No cloning theorem

Cannot measure qubits
during calculation

$$\mathbf{a}\,|\uparrow\rangle + \mathbf{b}\,|\downarrow\rangle \quad \to \quad |\uparrow\rangle, \; |\downarrow\rangle$$

yet there is hope!!

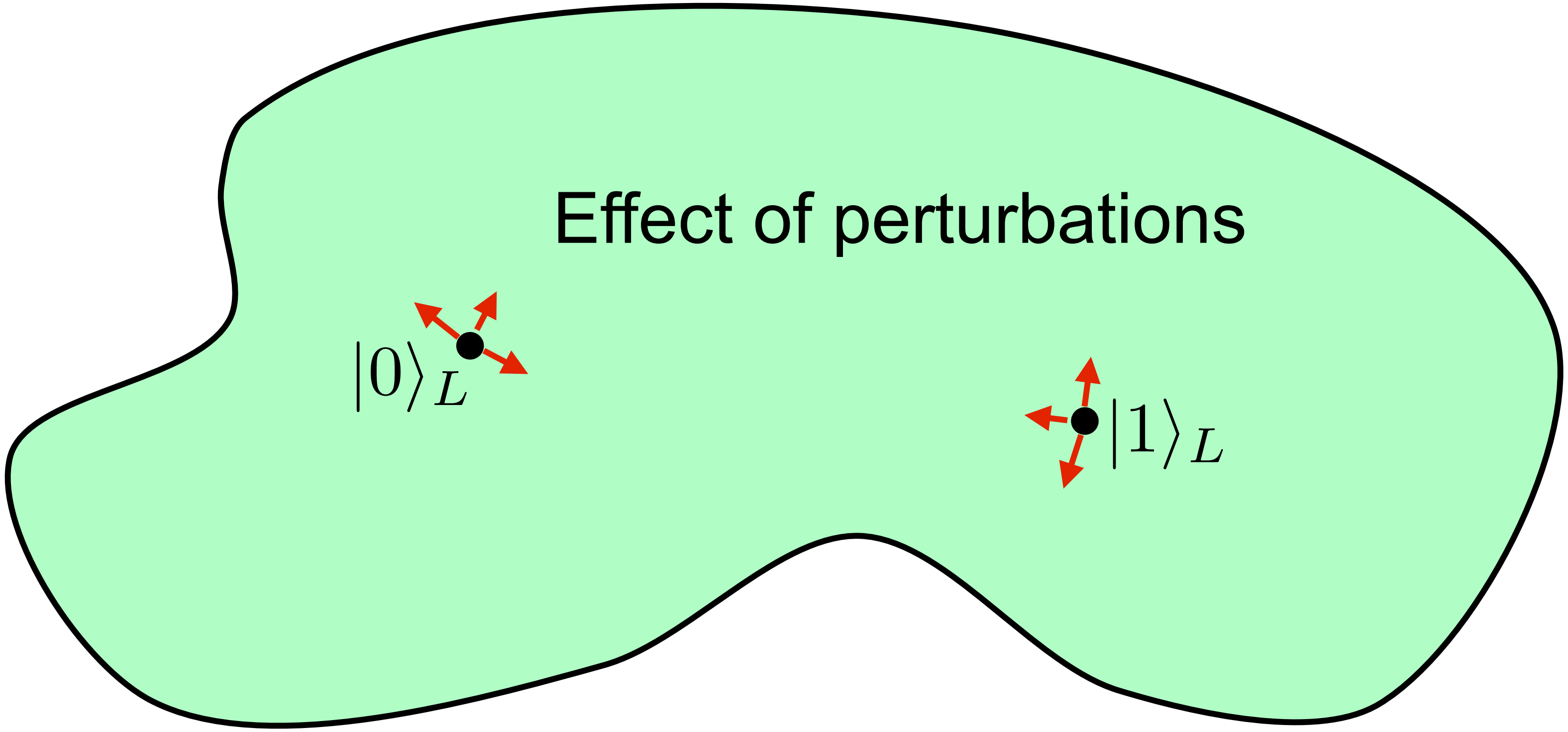
Threshold theorem

Quantum Error Correction

Challenges:

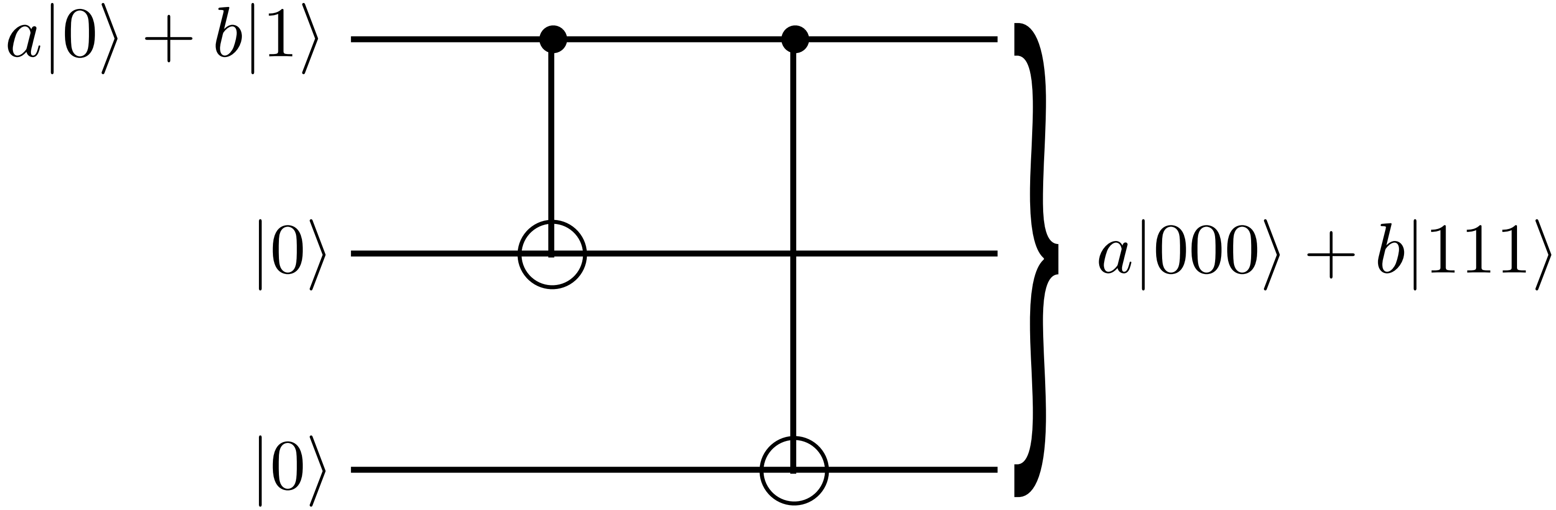
- Measurements must not perturb information
- Continuum of errors

"Redundancy" : Use larger Hilbert space



Error Correction : purge "forbidden" areas of Hilbert space

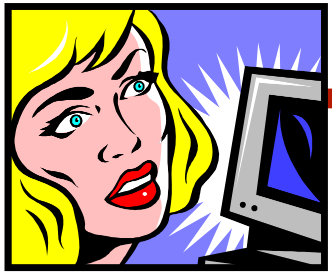
Encoding


$a|0\rangle + b|1\rangle$
$|0\rangle$
$|0\rangle$
$a|000\rangle + b|111\rangle$

Single Spin-Flip Error

for Quantum Communication

Alice



Bob

$|\Psi_0\rangle = a\,|0\rangle + b\,|1\rangle$

use 2 ancilla bits in state $|0\rangle$

$|\Psi_0\rangle = a\,|000\rangle + b\,|100\rangle$

CNOT^2

$|\Psi_1\rangle = a\,|000\rangle + b\,|111\rangle$

Possible error probability : p

Bit-flip on 2

$|\tilde{\Psi}_1\rangle = a\,|010\rangle + b\,|101\rangle$

$|\Psi_1\rangle = a\,|000\rangle + b\,|111\rangle$

Perfect transmission

Detection and Correction

$|\Psi_1\rangle = a\,|000\rangle + b\,|111\rangle$

$|\tilde{\Psi}_1\rangle = a\,|010\rangle + b\,|101\rangle$

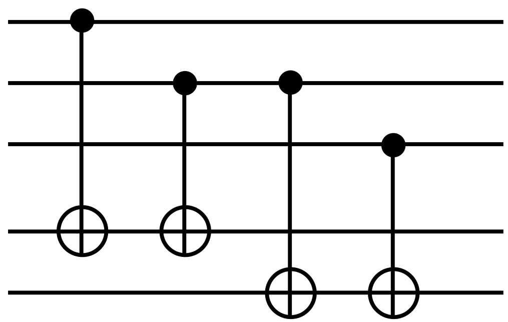
$\Big\}$ are eigenstates of Z_1Z_2 and Z_1Z_3

	Z_1Z_2	Z_1Z_3	
$a\,|000\rangle + b\,|111\rangle$	1	1	
$a\,|100\rangle + b\,|011\rangle$	-1	-1	flip 1 : X_1
$a\,|010\rangle + b\,|101\rangle$	-1	1	flip 2 : X_2
$a\,|001\rangle + b\,|110\rangle$	1	-1	flip 3 : X_3

Alternative error detection:

use 2 ancilla qubits in $|00\rangle$ state

$|00\rangle$



Excercises

1. **Qubit Projector in the Pauli Basis**

   Given a general single-qubit state:

   $$|\psi\rangle = a|0\rangle + b|1\rangle$$

   where a and b are complex numbers satisfying $|a|^2 + |b|^2 = 1$, express the projector $|\psi\rangle\langle\psi|$ as a linear combination of Pauli matrices:

   $$|\psi\rangle\langle\psi| = c_1 I + c_x X + c_y Y + c_z Z$$

Excercises

2. **Decoherence: Pure Dephasing from a Stochastic Phase**

Consider a spin-$\frac{1}{2}$ particle initially polarized along the x-axis, i.e.,

$$\rho(0) = \frac{1}{2}(I + S_x),$$

and subject to a time-dependent stochastic phase $\delta(t)$ accumulated due to a fluctuating magnetic field along the z-axis. That is, $\delta(t)$ incorporates the effect of the field over time, and represents the net phase difference acquired between the spin states $|0\rangle$ and $|1\rangle$. In the interaction (rotating) frame, the time evolution of the density matrix is given by

$$\rho(t) = e^{-i\delta(t)S_z}\,\rho(0)\,e^{i\delta(t)S_z}.$$

We are interested in computing the ensemble-averaged magnetization along the x-axis,

$$\langle S_x(t)\rangle = \mathrm{Tr}\left[\rho(t)S_x\right].$$

(a) Show that

$$\langle S_x(t)\rangle = \langle\cos(\delta(t))\rangle,$$

assuming that $\delta(t)$ is a real-valued stochastic variable with zero mean.

(b) Suppose $\delta(t)$ is a *Gaussian random variable* with zero mean and variance $\langle\delta^2(t)\rangle$. Express $\langle S_x(t)\rangle$ in terms of the real part of the characteristic function $\langle e^{i\delta(t)}\rangle$.

(c) Prove that for a zero-mean Gaussian random variable δ,

$$\langle e^{i\delta}\rangle = e^{-\langle\delta^2\rangle/2}.$$

Hint: Expand $e^{i\delta}$ in a Taylor series and use the known moments of a Gaussian variable: all odd moments vanish, and even moments are given by

$$\langle\delta^{2n}\rangle = (2n-1)!!\,\langle\delta^2\rangle^n.$$

(d) Use the result from part (c) to show that

$$\langle S_x(t)\rangle = e^{-\langle\delta^2(t)\rangle/2}.$$

(e) Consider the case where $\delta(t)$ arises from a random walk process (Markovian noise), such that

$$\langle\delta^2(t)\rangle = \frac{2t}{T_2},$$

i.e., the phase variance grows linearly with time. Show that

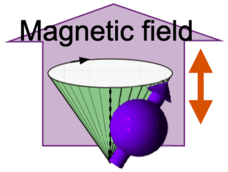
$$\langle S_x(t)\rangle = e^{-t/T_2},$$

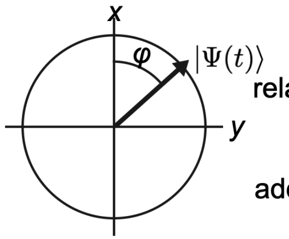
and interpret the physical meaning of the time constant T_2.

Semiclassical Description

$|\Psi(0)\rangle = a|0\rangle + b|1\rangle$

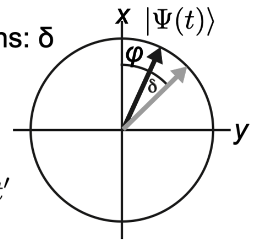
$|\Psi(t)\rangle = a|0\rangle e^{-i\mathcal{E}_0 t/\hbar} + b|1\rangle e^{-i\mathcal{E}_1 t/\hbar}$





relative phase: $\varphi = (\mathcal{E}_1 - \mathcal{E}_0)t/\hbar$

additional perturbations: δ



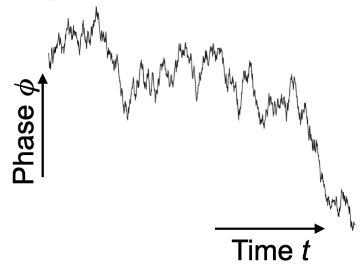
additional precession:

$$\delta(t) = \frac{1}{\hbar}\int_0^t (\delta_{\mathcal{E}_1} - \delta_{\mathcal{E}_0})dt'$$

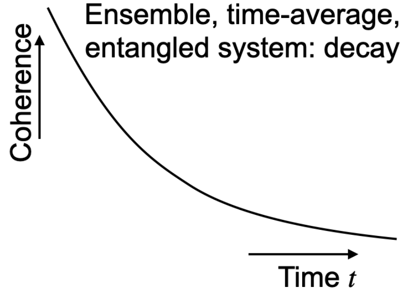
Time Dependence

The coupling is in general time dependent

Single qubit : diffusion process



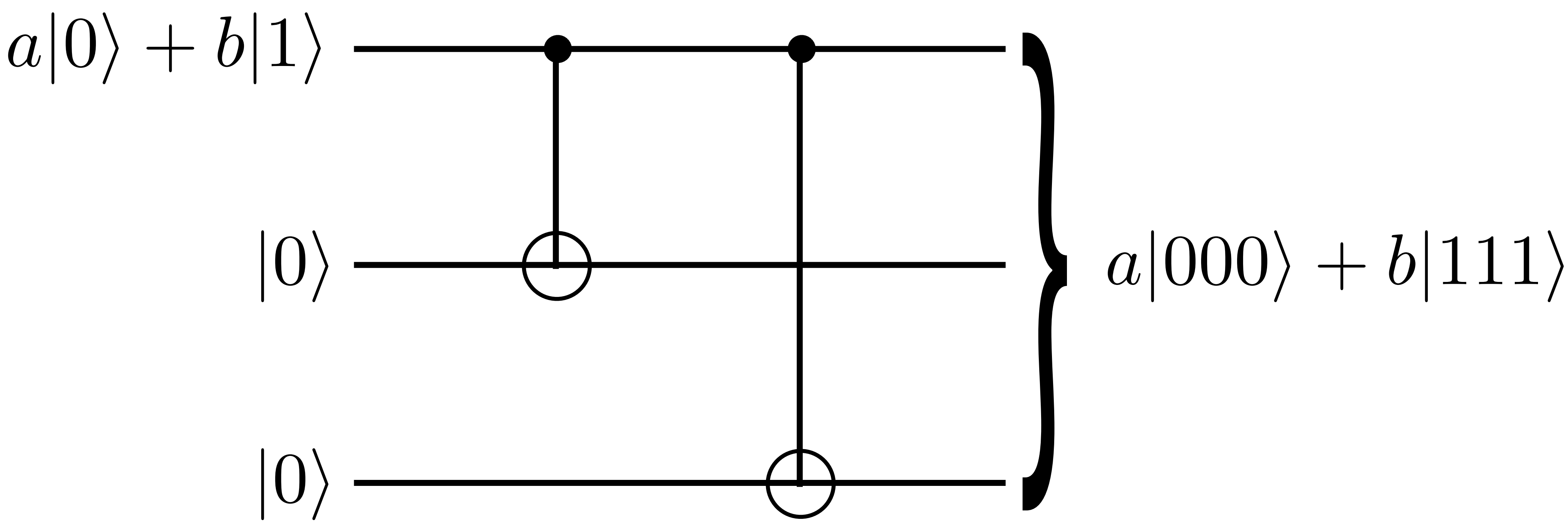
Ensemble, time-average, entangled system: decay



The observed polarization therefore decays

$$\rho_{ij}(t) = \rho_{ij}(0)\,e^{-i(\mathcal{E}_i - \mathcal{E}_j)t/\hbar}e^{-t/T_2}$$

Encoding


$a|0\rangle + b|1\rangle$
$|0\rangle$
$|0\rangle$
$a|000\rangle + b|111\rangle$

Single Spin-Flip Error

for Quantum Communication

Alice

Bob

$|\Psi_0\rangle = a\,|0\rangle + b\,|1\rangle$

use 2 ancilla bits in state $|0\rangle$

$|\Psi_0\rangle = a\,|000\rangle + b\,|100\rangle$

CNOT^2

Possible error probability : p

$|\Psi_1\rangle = a\,|000\rangle + b\,|111\rangle \xrightarrow{\text{Bit-flip on 2}} |\tilde{\Psi}_1\rangle = a\,|010\rangle + b\,|101\rangle$

$|\Psi_1\rangle = a\,|000\rangle + b\,|111\rangle \xrightarrow[\text{Perfect transmission}]{} |\Psi_1\rangle = a\,|000\rangle + b\,|111\rangle$

Detection and Correction

$|\Psi_1\rangle = a\,|000\rangle + b\,|111\rangle$
$|\tilde{\Psi}_1\rangle = a\,|010\rangle + b\,|101\rangle$
$\Big\}$ are eigenstates of Z_1Z_2 and Z_1Z_3

	Z_1Z_2	Z_1Z_3		
$a\,|000\rangle + b\,|111\rangle$	1	1		
$a\,|100\rangle + b\,|011\rangle$	-1	-1	$\Rightarrow$	flip 1 : X_1
$a\,|010\rangle + b\,|101\rangle$	-1	1	$\Rightarrow$	flip 2 : X_2
$a\,|001\rangle + b\,|110\rangle$	1	-1	$\Rightarrow$	flip 3 : X_3

Continuous Phase Errors

Error : random z-rotation

$$P(\varepsilon) : e^{i\varepsilon\phi \mathbf{Z}} = \begin{pmatrix} e^{i\varepsilon\phi} & 0 \\ 0 & e^{-i\varepsilon\phi} \end{pmatrix} = \cos(\varepsilon\phi)\mathbf{1} + i\sin(\varepsilon\phi)\mathbf{Z}$$

initial state

$|\Psi(0)\rangle = a\,|0\rangle + b\,|1\rangle$

$a\,|0\rangle + b\,|1\rangle$

$a\,|0\rangle - b\,|1\rangle$

"Phase flip"

use H to change basis

x-basis

$|\Psi(0)\rangle = |\pm_X\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

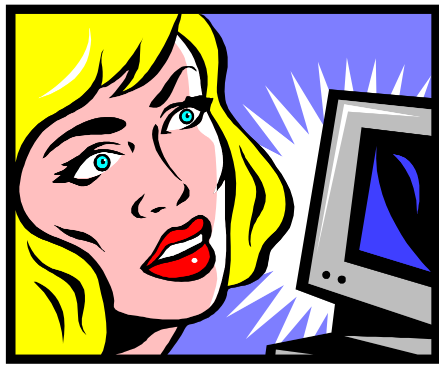
$\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

$\frac{1}{\sqrt{2}}(|0\rangle \mp |1\rangle)$

"Bit flip"

Continuous Phase Error

Alice



Bob

$|\Psi_0\rangle = a\,|0\rangle + b\,|1\rangle$

$|\Psi_0\rangle = a\,|000\rangle + b\,|100\rangle$

$\mathrm{CNOT}^{\otimes 2}$

$|\Psi_1\rangle = a\,|000\rangle + b\,|111\rangle$

$\mathrm{H}^{\otimes 3}$

$|\Psi_1\rangle = a\,|{+}{+}{+}\rangle + b\,|{-}{-}{-}\rangle$

Phase error (Z_2)

$|\widetilde{\Psi}_1\rangle = a\,|{+}{-}{+}\rangle + b\,|{-}{+}{-}\rangle$

$|\Psi_1\rangle = a\,|{+}{+}{+}\rangle + b\,|{-}{-}{-}\rangle$

Check by X_1X_2, X_1X_3

Error correction by Z_1, Z_2, Z_3

Switch back to comp. basis by $\mathrm{H}^{\otimes 3}$

Projection Errors

~ accidental measurement $\quad a\,|\!\uparrow\rangle + b\,|\!\downarrow\rangle$ $\rightarrow$ $|\!\uparrow\rangle$, $|\!\downarrow\rangle$

Projection operators:

$$|0\rangle\langle 0| = P_\uparrow = \tfrac{1}{2}(1+Z) \qquad |1\rangle\langle 1| = P_\downarrow = \tfrac{1}{2}(1-Z)$$

Projection errors can be corrected like phase errors

Any single qubit error can be expanded in the basis **1, X, Y, Z**

"Discretization of errors"

Excercise

General single-qubit state $|\Psi\rangle = a|0\rangle + b|1\rangle$

Write the projector onto $|\Psi\rangle$ as a linear combination

$$|\Psi\rangle\langle\Psi| = c_1\mathbf{1} + c_x\mathbf{X} + c_y\mathbf{Y} + c_z\mathbf{Z}$$

Arbitrary Single Qubit Errors

Can be corrected by concatenating schemes for bit flip (**X**) and phase errors (**Z**)

1) 3-qubit code $\quad |0\rangle \rightarrow |000\rangle\,; \quad |1\rangle \rightarrow |111\rangle \qquad$ **CNOT**$^{\otimes 2}$

2) switch to *x*-base

$$\mathbf{H}^{\otimes 3} : |000\rangle \rightarrow |{+}{+}{+}\rangle\,; \quad |111\rangle \rightarrow |{-}{-}{-}\rangle$$

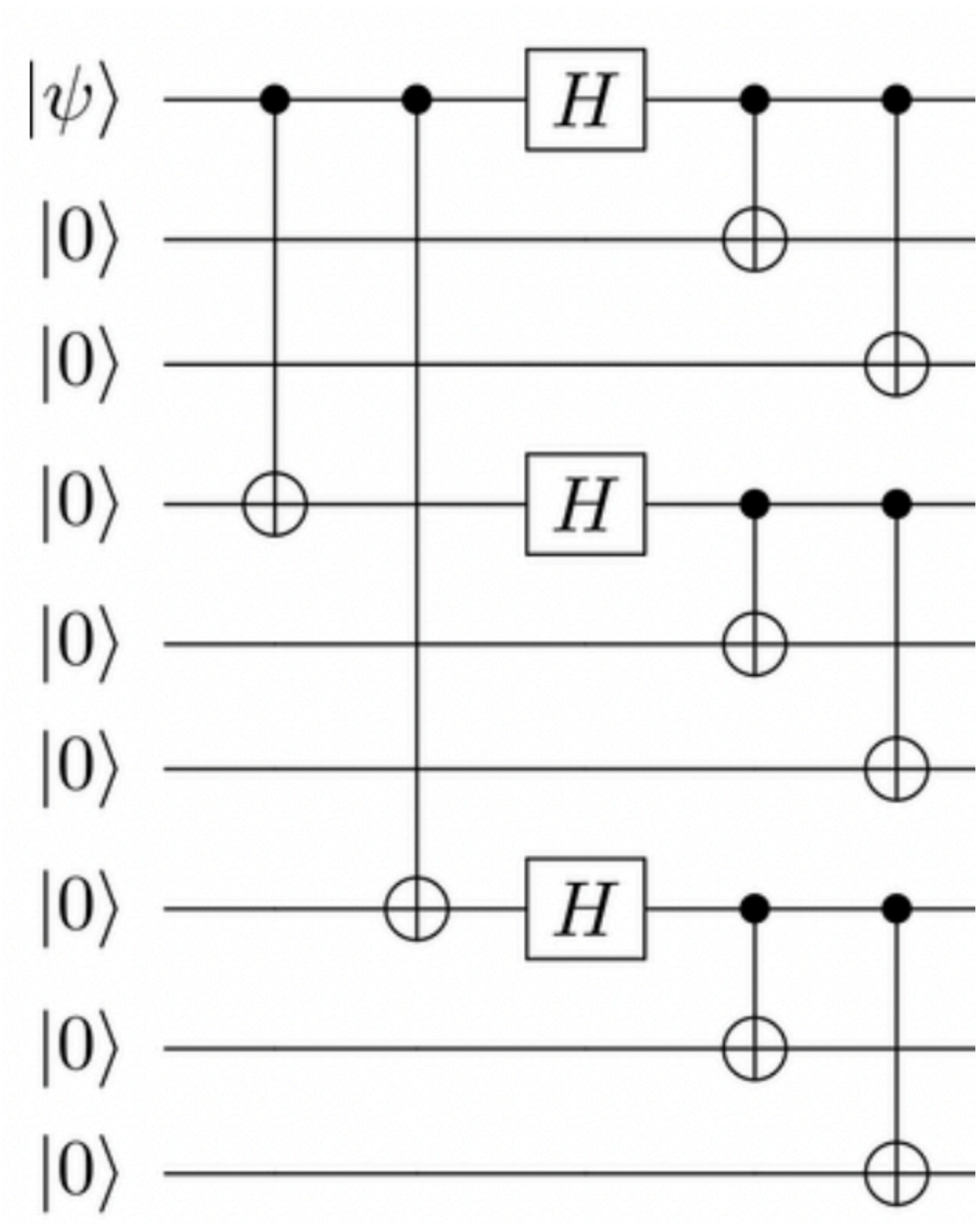
$|00\rangle$ $\qquad$ **CNOT**$^{\otimes 2}$

3) Encode each qubit again into 3

$$|0\rangle \rightarrow \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|1\rangle \rightarrow \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)\,(|000\rangle - |111\rangle)\,(|000\rangle - |111\rangle)$$

Arbitrary SingleQE - 9 qubits



Syndrome Extraction

$$|0\rangle \rightarrow \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|1\rangle \rightarrow \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

1) Check for bit-flip errors using Z_1Z_2, Z_1Z_3 etc.

Correct (if necessary) using X_1 .. X_9.

2) Phase errors: $(|000\rangle+|111\rangle \rightarrow |000\rangle-|111\rangle)$

detect by comparing signs of blocks

using $X_1X_2X_3X_4X_5X_6$, $X_1X_2X_3X_7X_8X_9$ etc.

Correct (if necessary) using $Z_1Z_2Z_3$ etc.

Concatenation corrects automatically $Y = XZ$ errors

Arbitrary Single Qubit Errors

9-bit code (see above):

P.W. Shor, 'Scheme for reducing decoherence in quantum computer memory', Phys. Rev. A 52, R2493 (1995).

Other encoding schemes

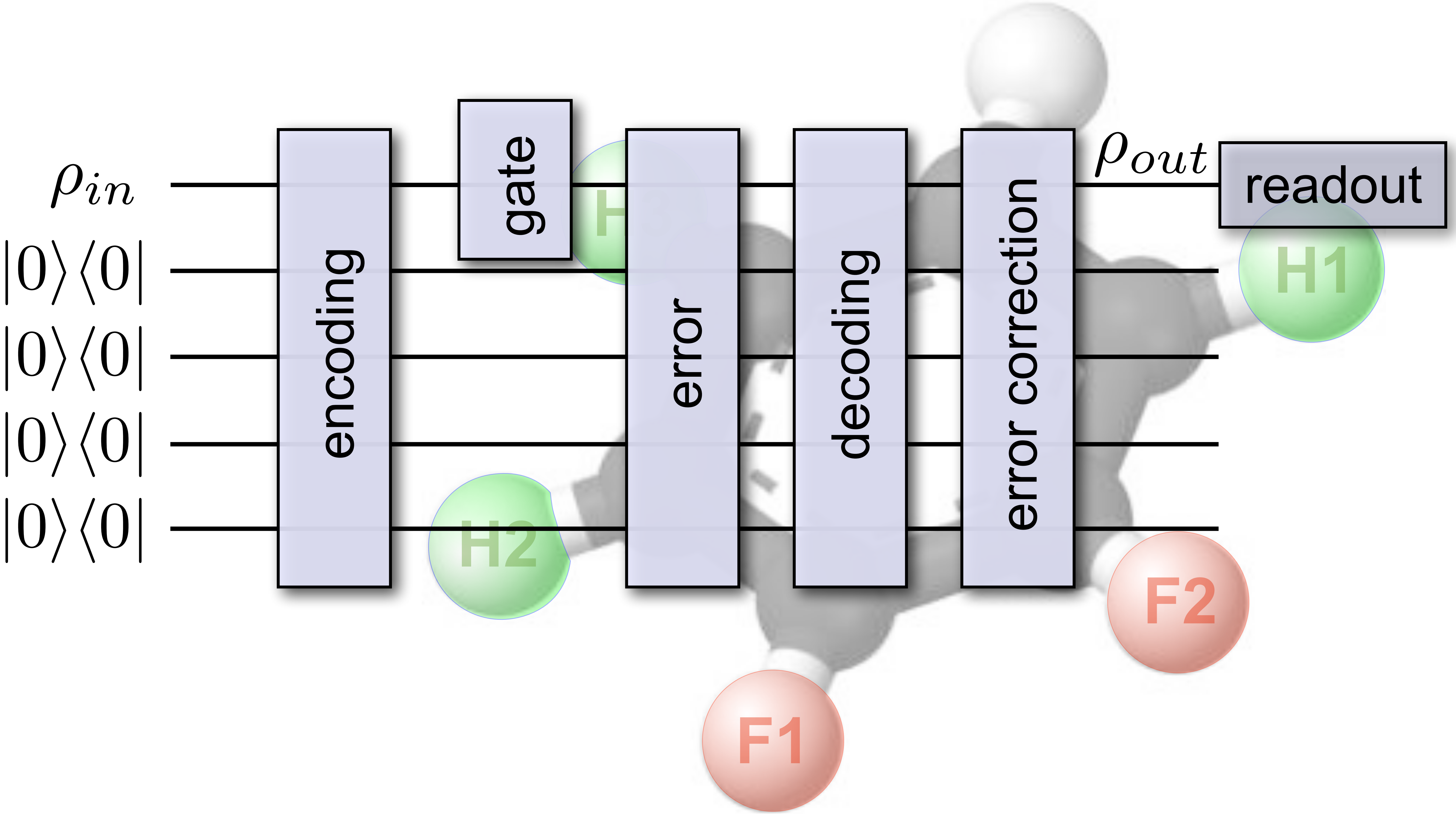
7-bit code:

A.M. Steane, 'Error Correcting Codes in Quantum Theory', Phys. Rev. Lett. 77, 793 (1996).

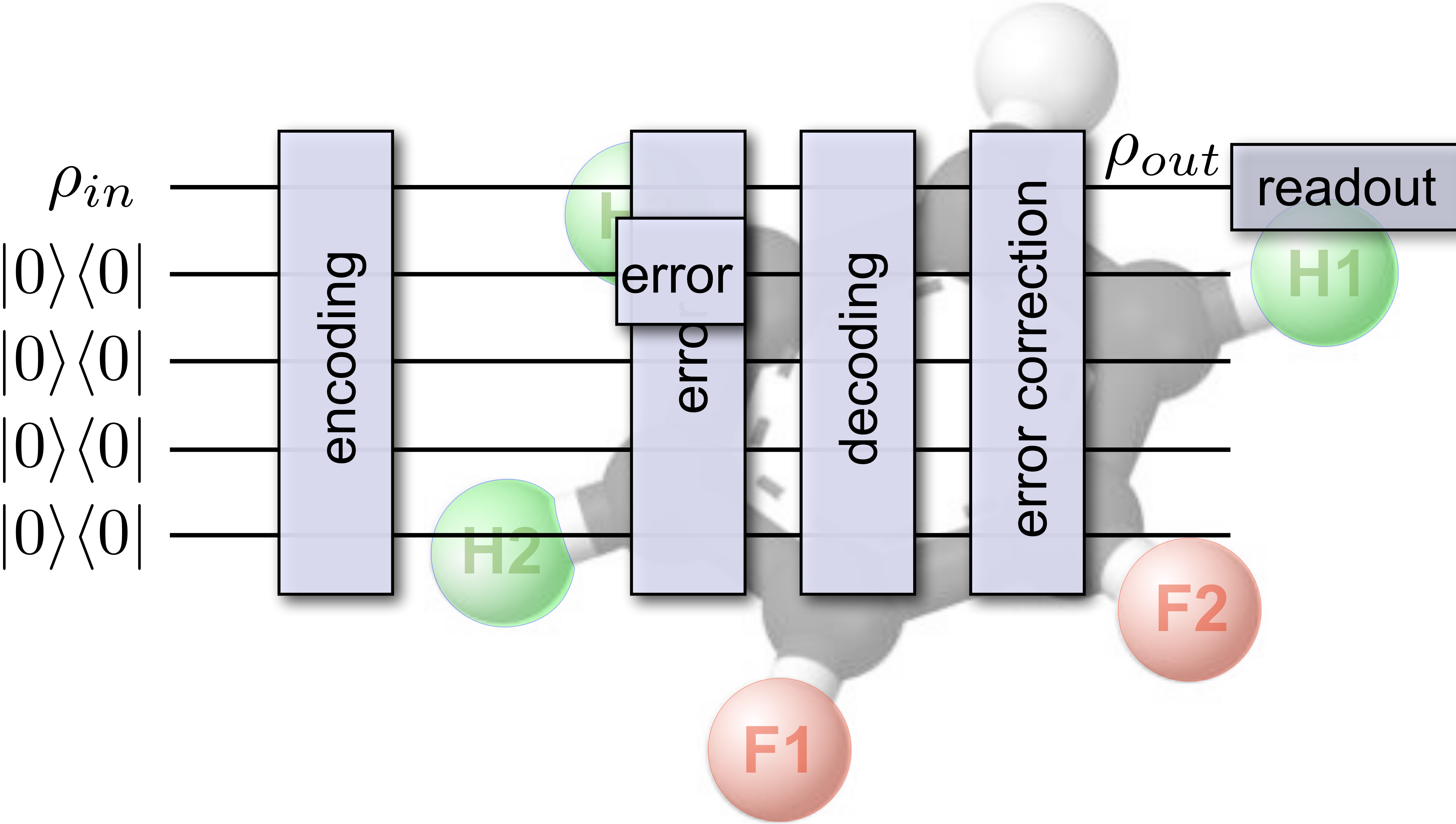
5-bit code:

R. Laflamme, C. Miquel, J.P. Paz, and W.H. Zurek, 'Perfect Quantum Error Correcting Code', Phys. Rev. Lett. 77, 198 (1996).
C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, 'Mixed-state entanglement and quantum error correction', Phys. Rev. A 54, 3824 (1996).

Quantum Error Correction



Quantum Error Correction



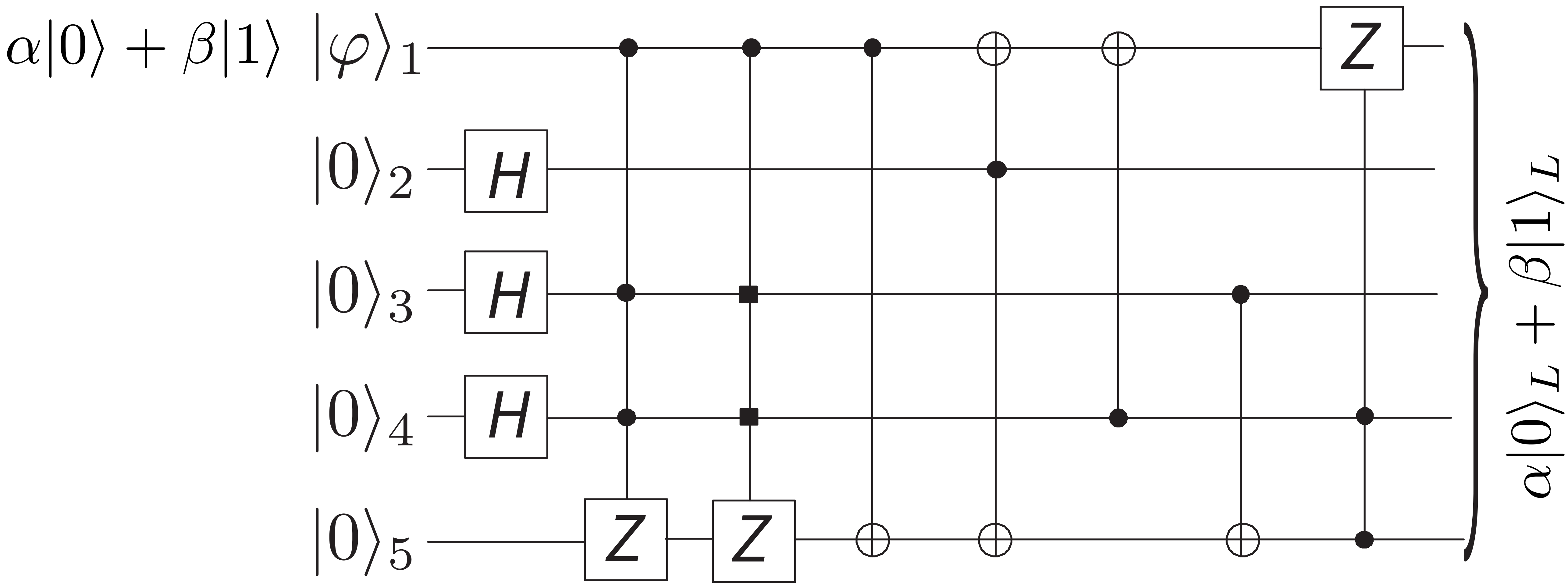
5-Qubit Encoding

Encoded basis states:

$$|0\rangle_L = \frac{1}{\sqrt{8}}(|00000\rangle - |10111\rangle - |01011\rangle + |11100\rangle + |10010\rangle + |00101\rangle + |11001\rangle + |01110\rangle)$$

$$|1\rangle_L = \frac{1}{\sqrt{8}}(|11111\rangle - |01000\rangle + |10100\rangle - |00011\rangle + |01101\rangle + |11010\rangle - |00110\rangle - |10001\rangle).$$

Encoding scheme:



Groups

- Closure: For any pair of group elements $a, b \in G$, the result of the group operation must be in the group, $a \cdot b \in G$.

- Associativity: For all group elements $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

- Identity element: The group contains an element called identity and often written as $\mathbf{1}$, the the group operation yields $\mathbf{1} \cdot a = a \cdot \mathbf{1} = a$.

- Inverse element: For every group element a, there is an inverse element a^{-1}, such that $a \cdot a^{-1} = a^{-1} \cdot a = \mathbf{1}$.

Stabilizer Codes

Pauli group for 1 qubit: $\{\pm 1, \pm X, \pm Y, \pm Z, \pm i1, \pm iX, \pm iY, \pm iZ\}$

Subgroup S, e.g. (for 3 qubits) S = $\{1, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$

stabilizer

Set of states invariant under all elements of S

stabilized by S **e.g. V_S = |000>, |111>**

Elements of S are syndrome extractors

Codes characterized by [n, k, d]

n : # physical bits

k : # logical bits

d : minimum distance

(# differing qubits)

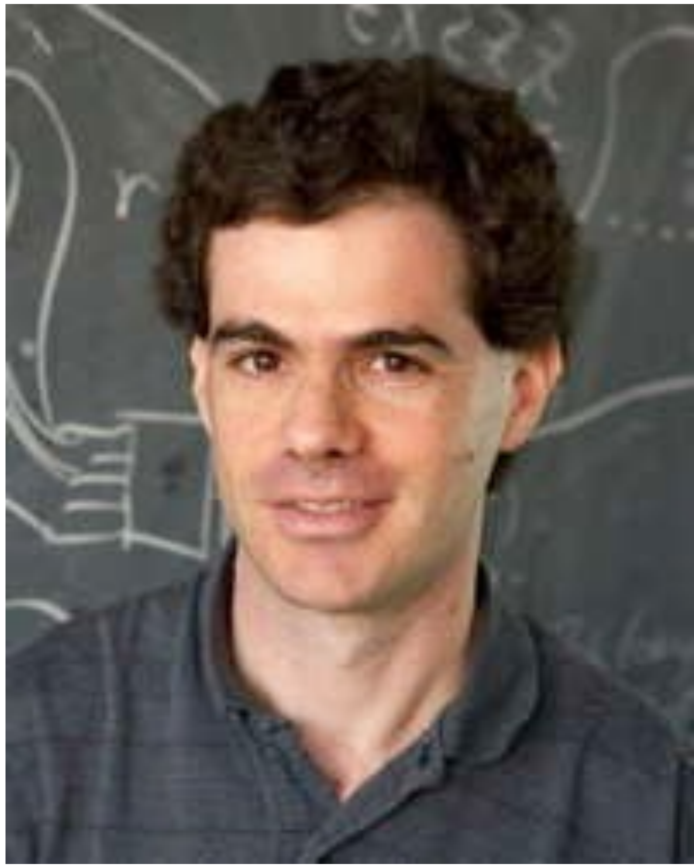
Quantum Error Correction Sonnet

We cannot clone, perforce; instead, we split
Coherence to protect it from that wrong
That would destroy our valued quantum bit
And make our computation take too long.

Correct a flip and phase - that will suffice.
If in our code another error's bred,
We simply measure it, then God plays dice,
Collapsing it to X or Y or Zed.

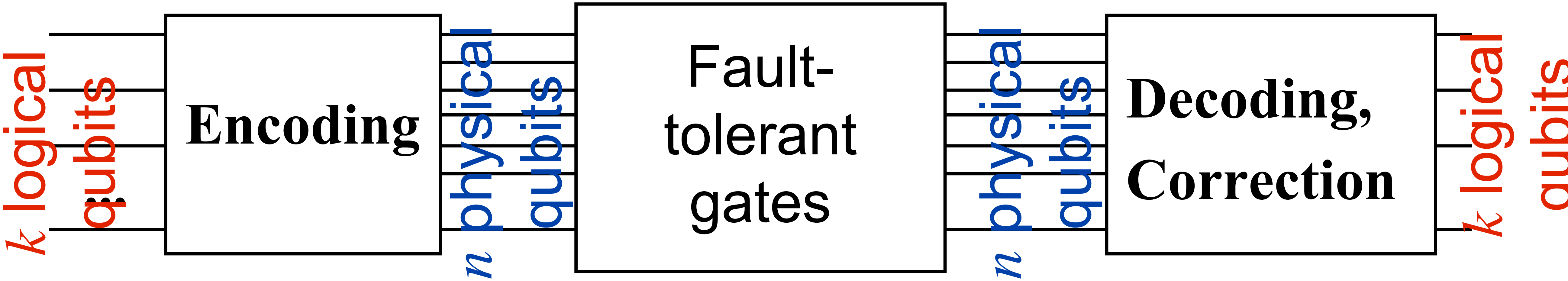
We start with noisy seven, nine, or five
And end with perfect one. To better spot
Those flaws we must avoid, we first must strive
To find which ones commute and which do not.

With group and eigenstate, we've learned to fix
Your quantum errors with our quantum tricks.



Daniel Gottesman

Fault-Tolerant Computing

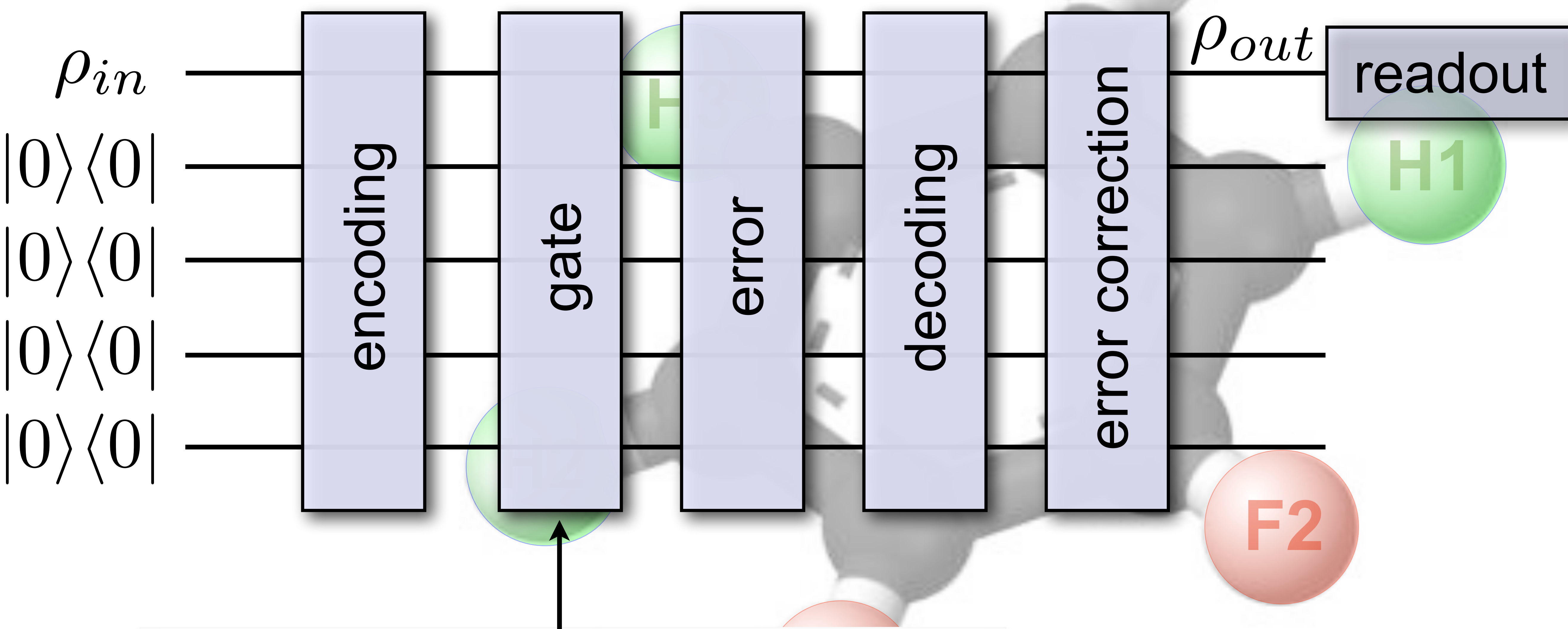


Gate operations must operate on full redundant code words, not only logical qubits

Quantum Error Correction

QEC for quantum computing

Is QEC compatible with processing ?

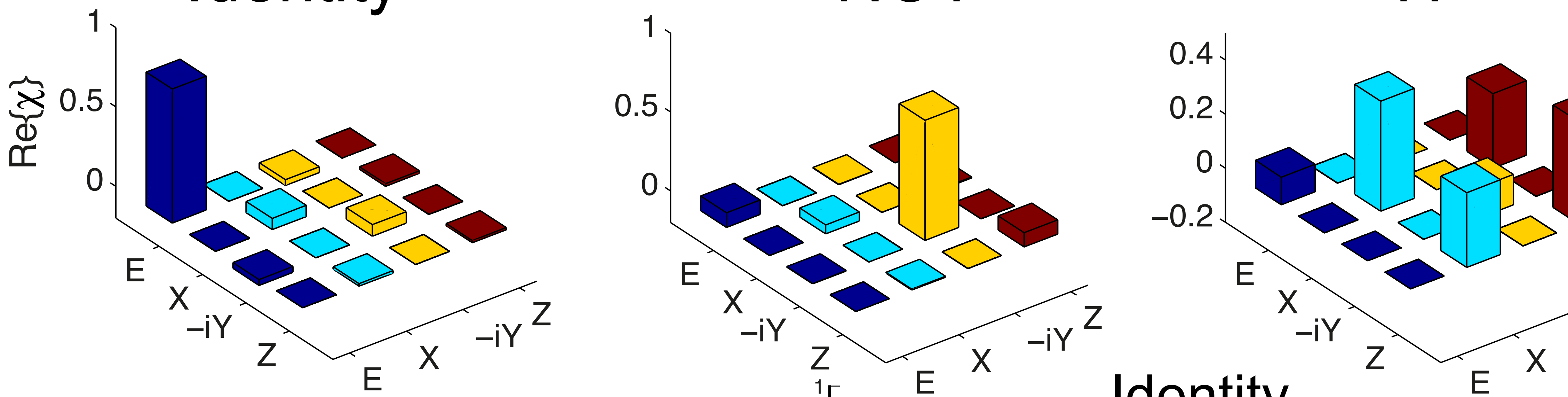


must be applied to logical states

5-qubit gate in the space of physical qubits

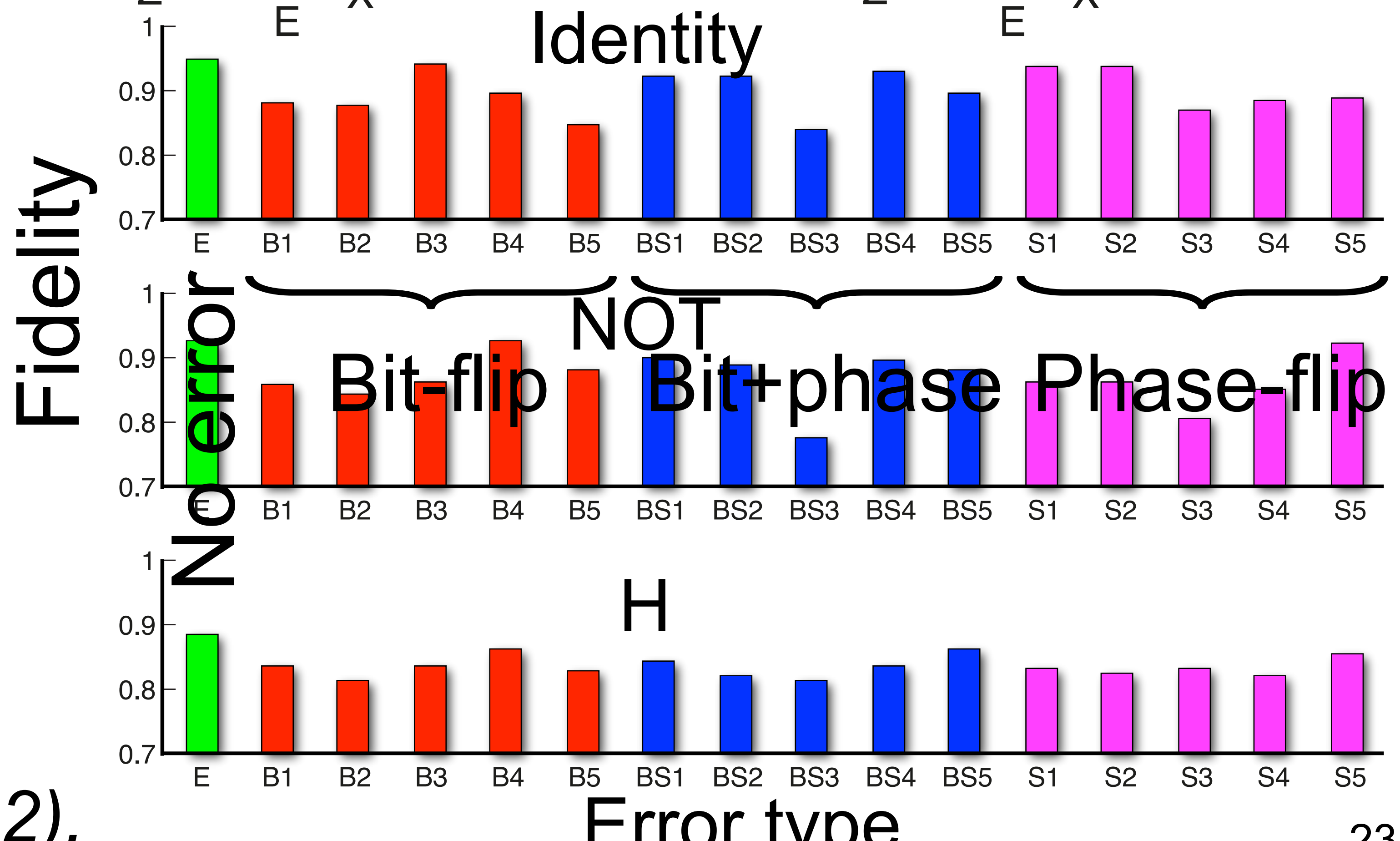
Quantum Error Correction

Process tomography of encoded gates



Individual results:

16 possible outcomes:
NoErr
3·5 = 15 different errors



Phys. Rev. Lett. <u>109</u>:100503 (2012).

Threshold Theorem

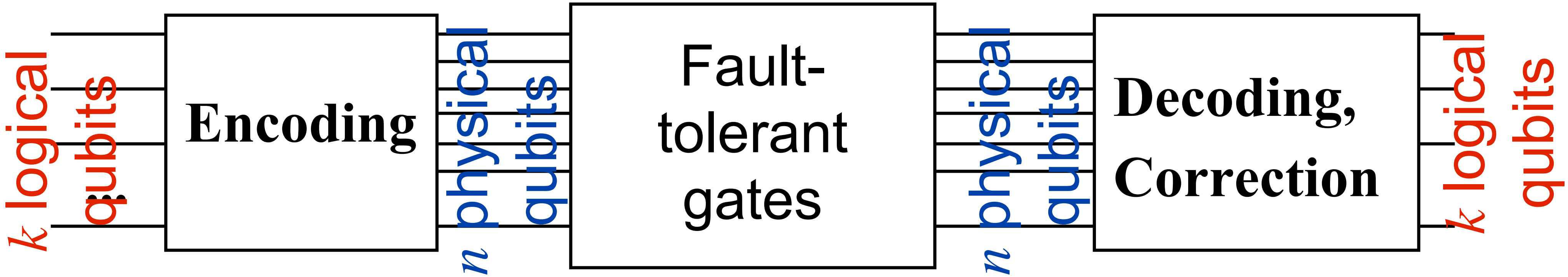
A quantum computation can be as long as required with any desired accuracy as long as the noise level is below a threshold value

J. Preskill, 'Reliable quantum computers', Proc. R. Soc. Lond. A 454, 385 (1998).

E. Knill. Quantum computing with realistically noisy devices. Nature 434, 39 (2005).

P. Aliferis, D. Gottesman, and J. Preskill. Accuracy threshold for postselected quantum computation. Quantum Information and Computation 8, 181 (2008).

Fault-Tolerant Computing



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Quantum Error Correction

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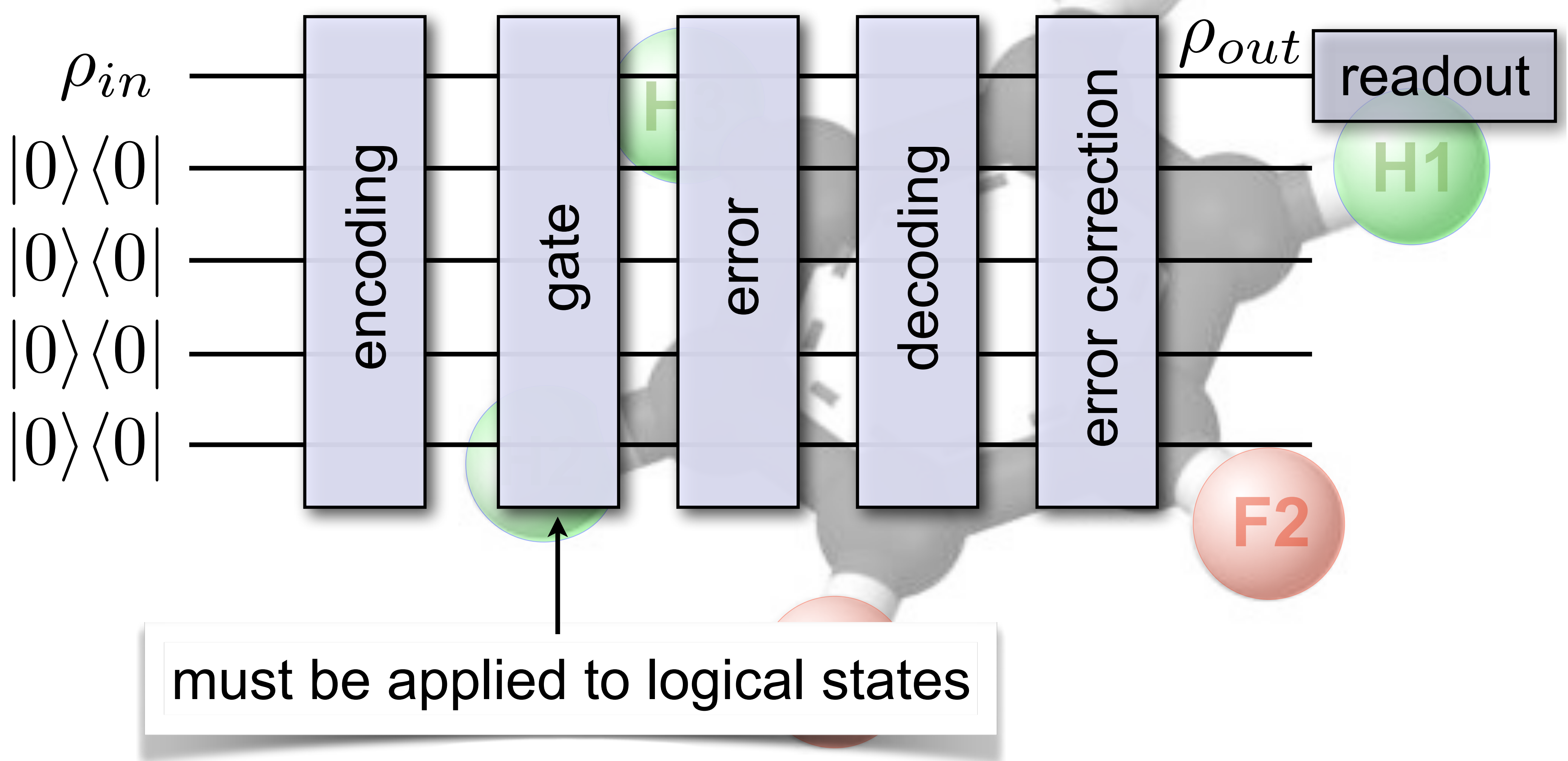
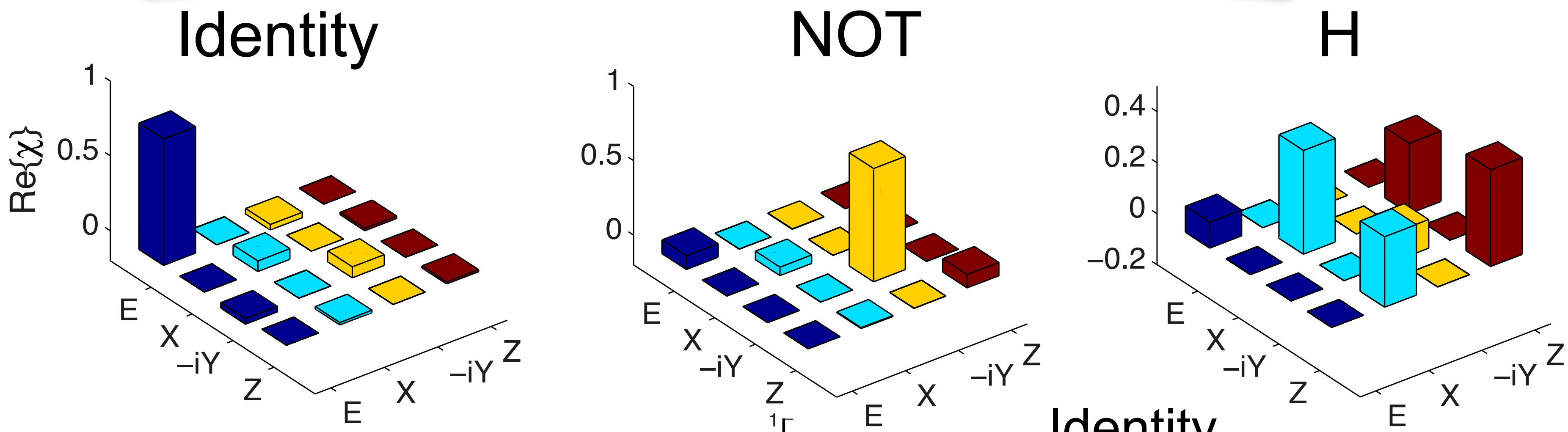


5-qubit gate in the space of physical qubits

Quantum Error Correction

Process tomography of encoded gates



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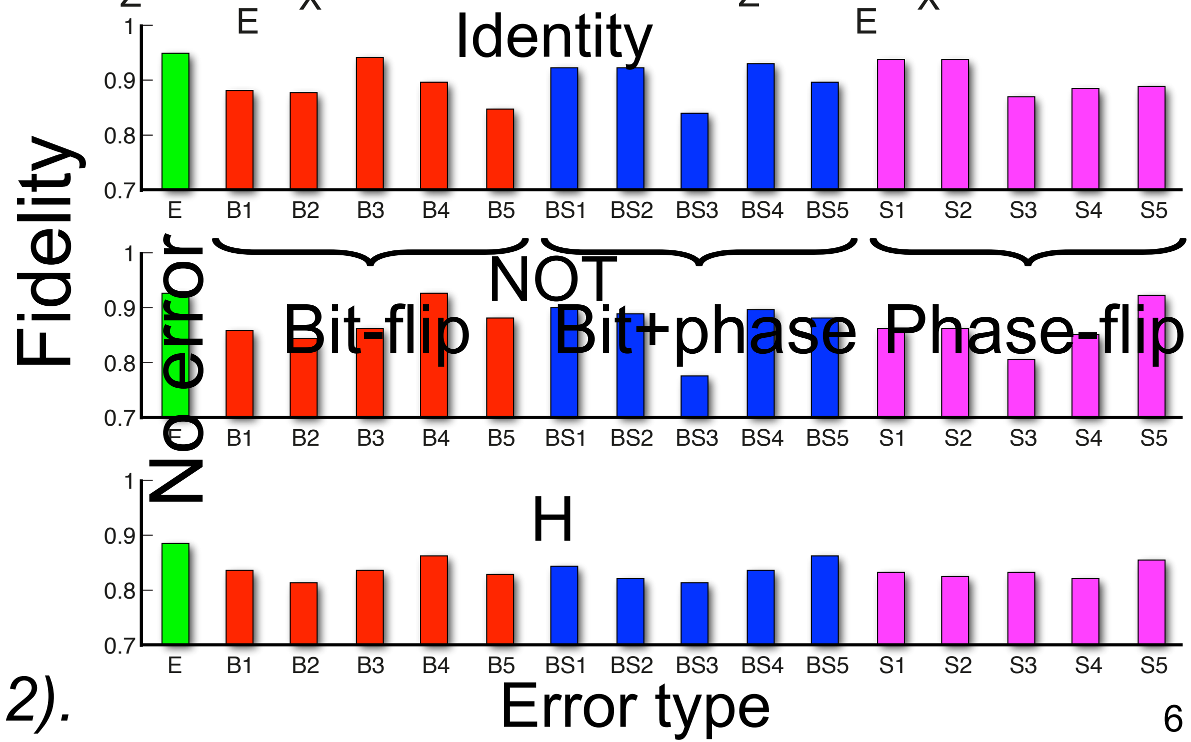

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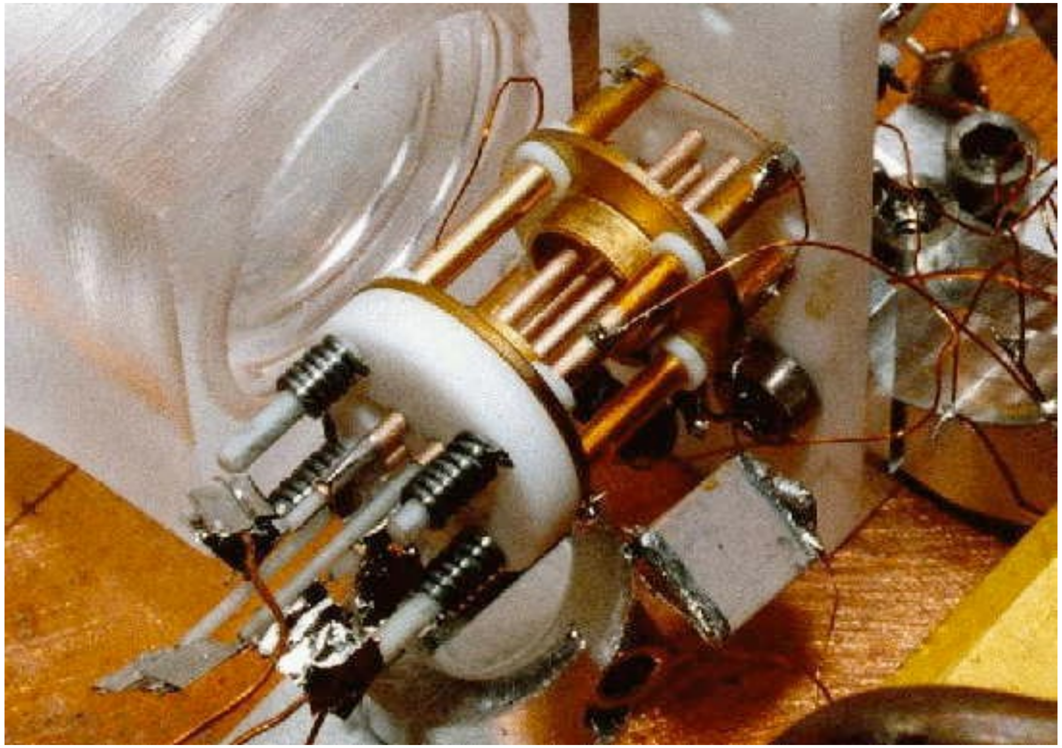
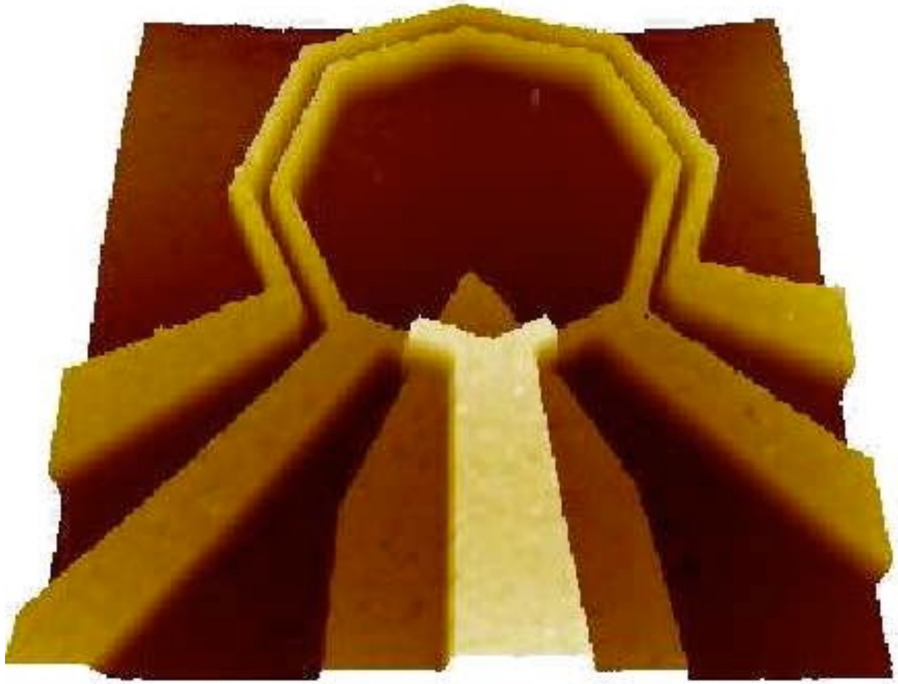
7.5 Avoiding Errors

Sources of Errors

- Spurious interactions : internal, environment



Ion trap

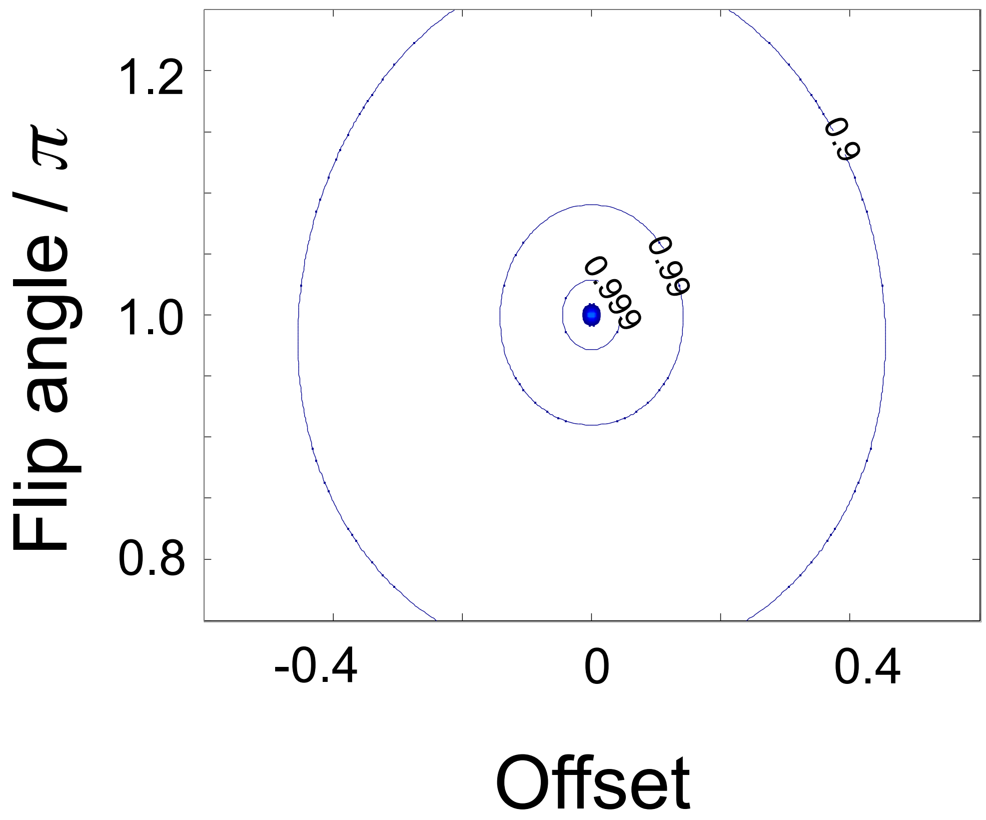


SQUID

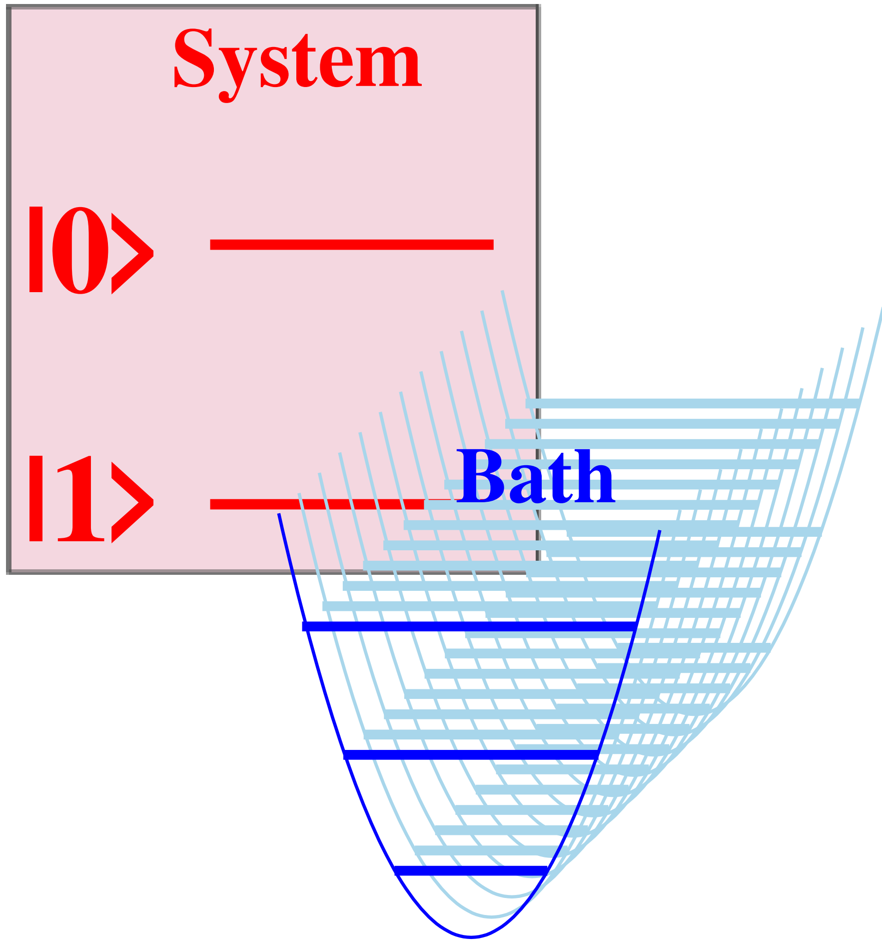
Sources of Errors

Imperfect gate operations

Fidelity of real laser pulse



System-environement interations



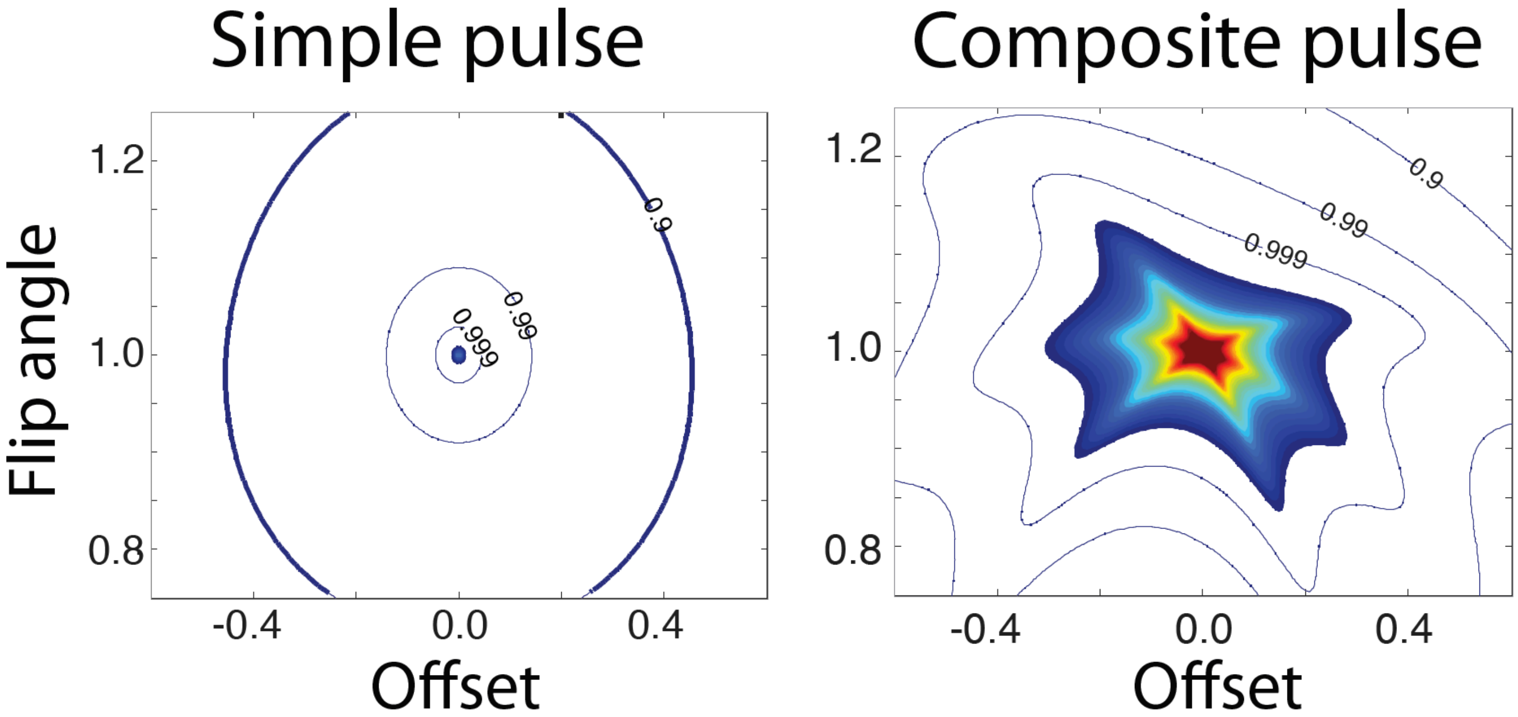
Fighting Errors

Threshold must be reached: 10^{-2} .. 10^{-4}

- Optimize the classical apparatus that controls the quantum system to make the gate operations as perfect as possible.
- Design gate operations in such a way that errors in experimental parameters tend to cancel rather than amplify.
- Use error correction schemes.
- Store the information in areas of the Hilbert space that are least affected by the interaction between the system and its environment.
- Use active schemes for decoupling the system from the environment, such as dynamical decoupling.

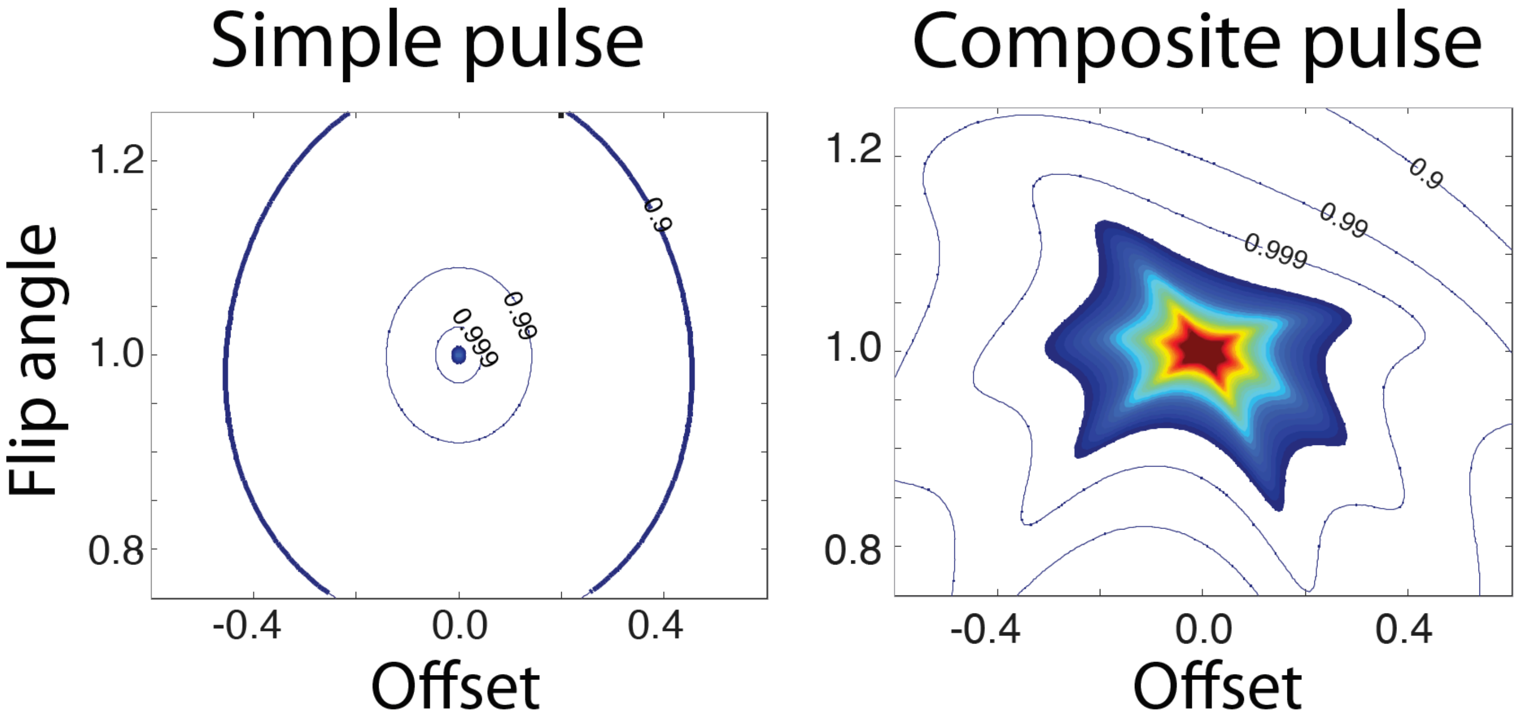
Counter-Strategy

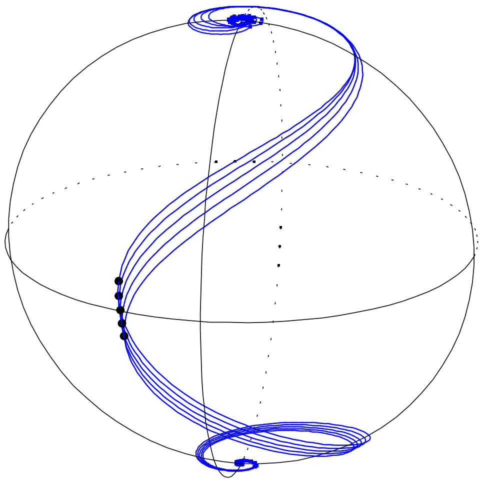
Design gate operations such that errors in experimental parameters cancel rather than amplify each other



Counter-Strategy

Design gate operations such that errors in experimental parameters cancel rather than amplify each other



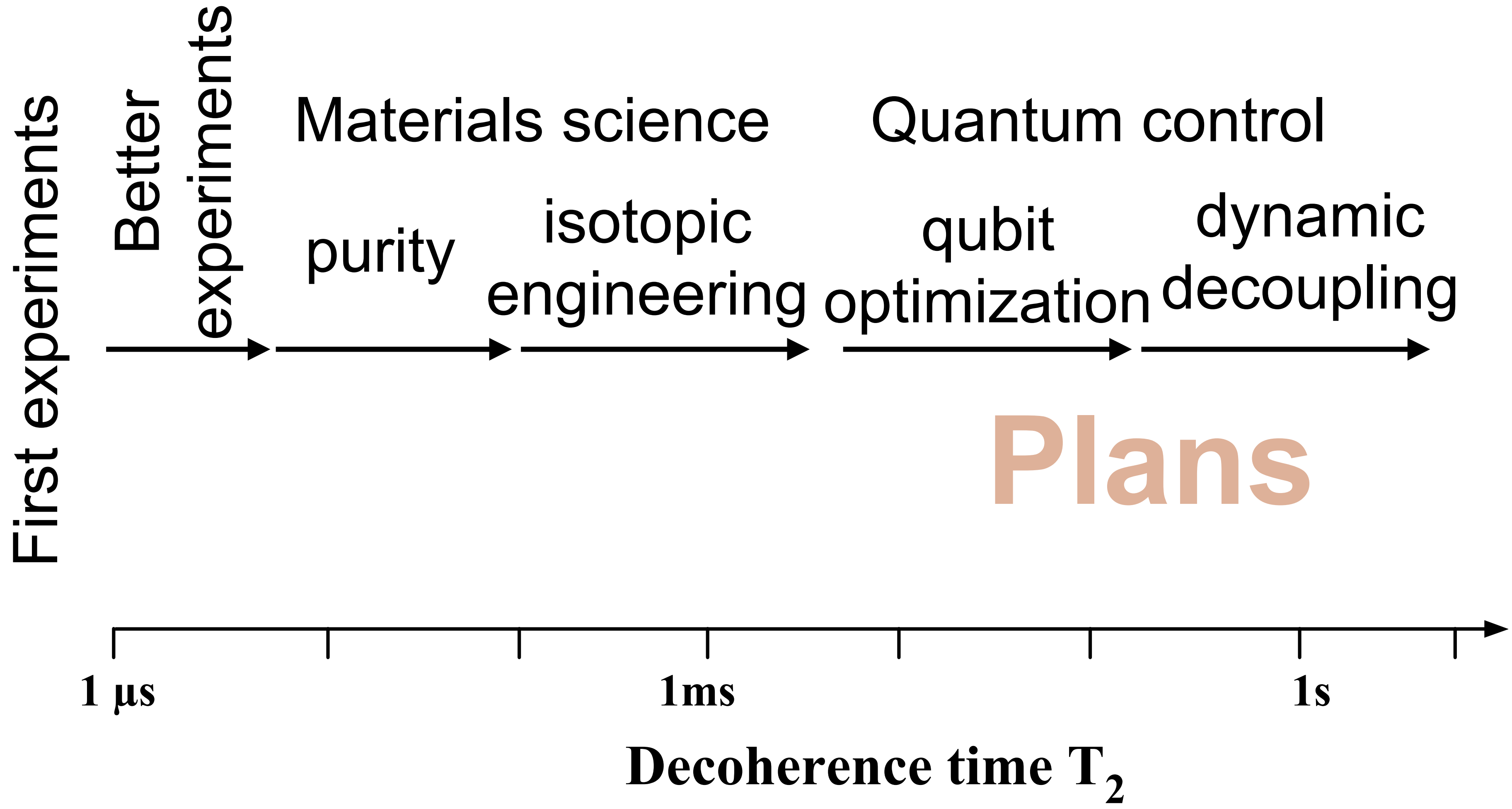


hyperbolic
secant pulse

Combination Therapy

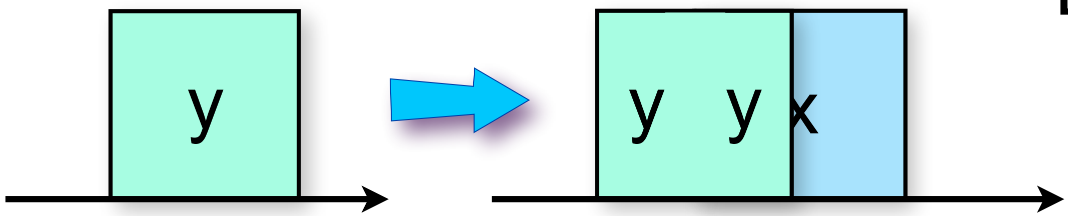
General principle : need to combine all possible countermeasures

Specific example : decoherence in diamond NV centers



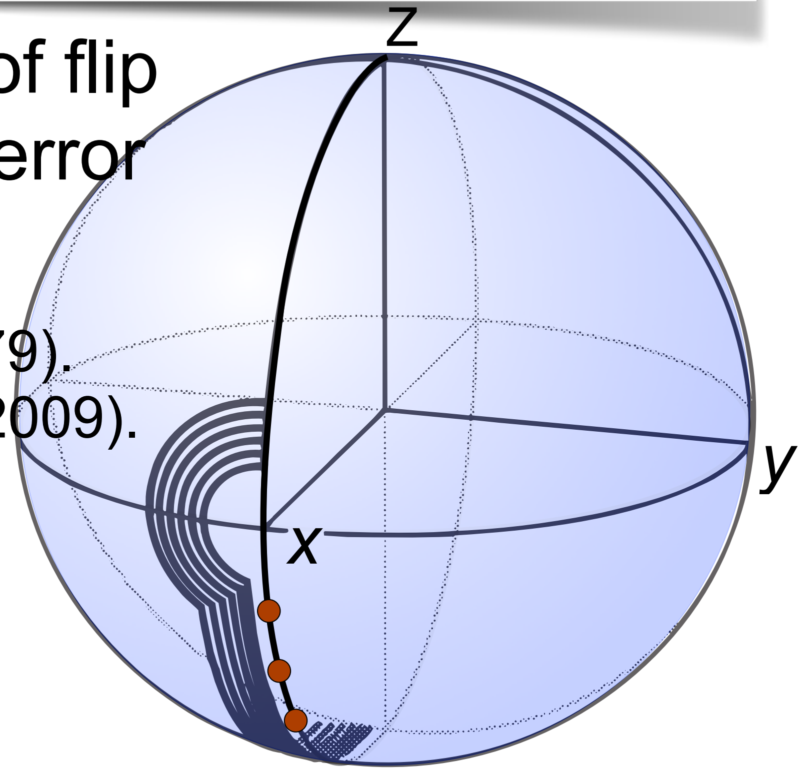
Error Compensation

Composite pulses = robust pulses = compensated pulses

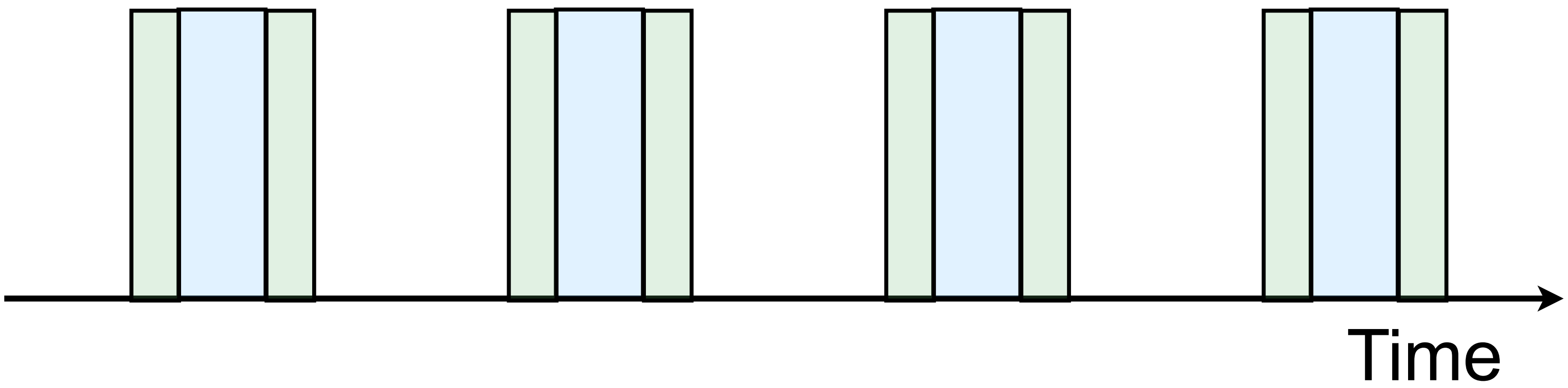


Levitt and Freeman, J. Magn. Reson. 33, 473 (1979).
Khodjasteh and Viola, Phys. Rev. A, 80, 032314 (2009).

Effect of flip angle error

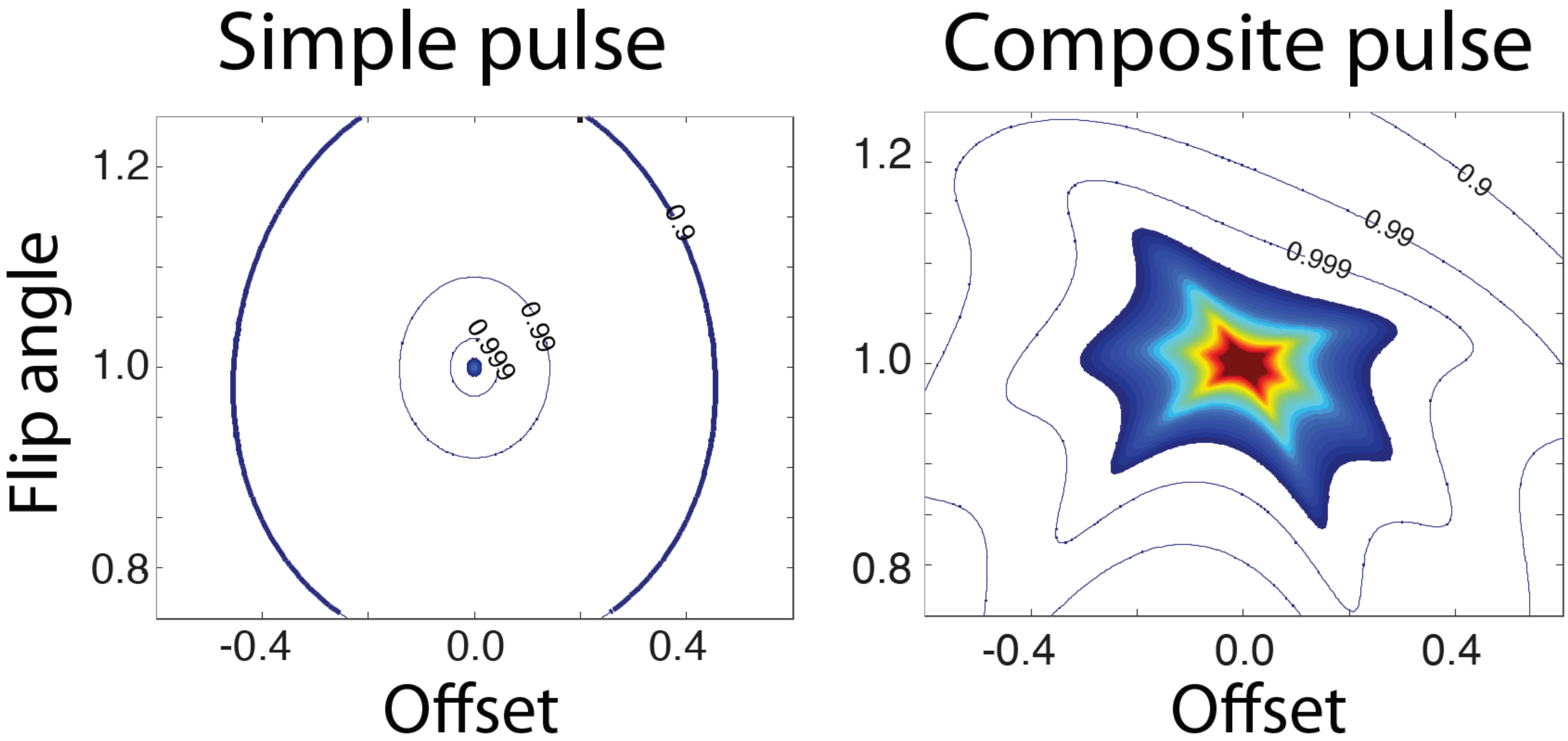


Making DD sequences robust:

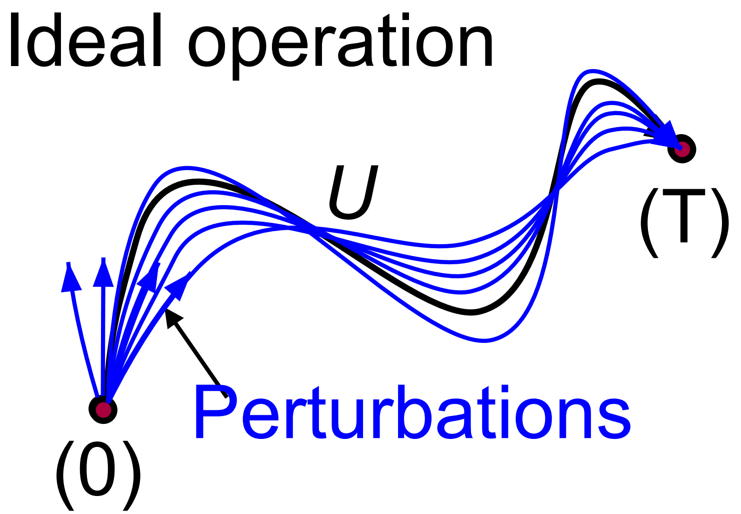


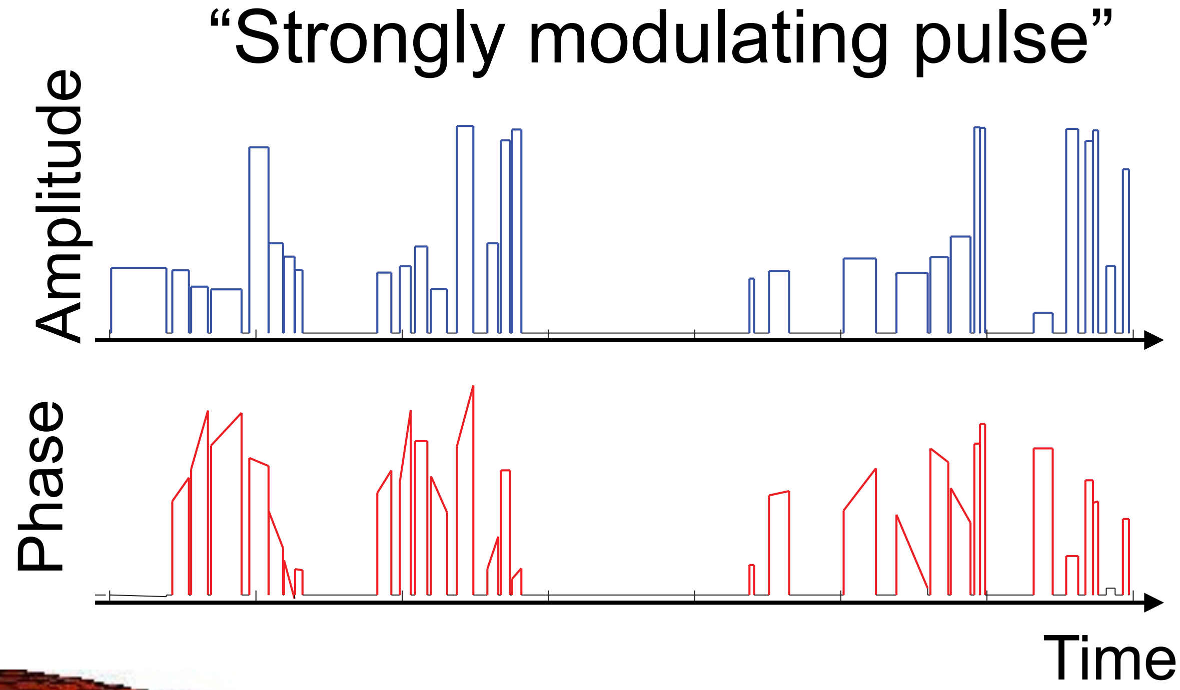
Error Compensation

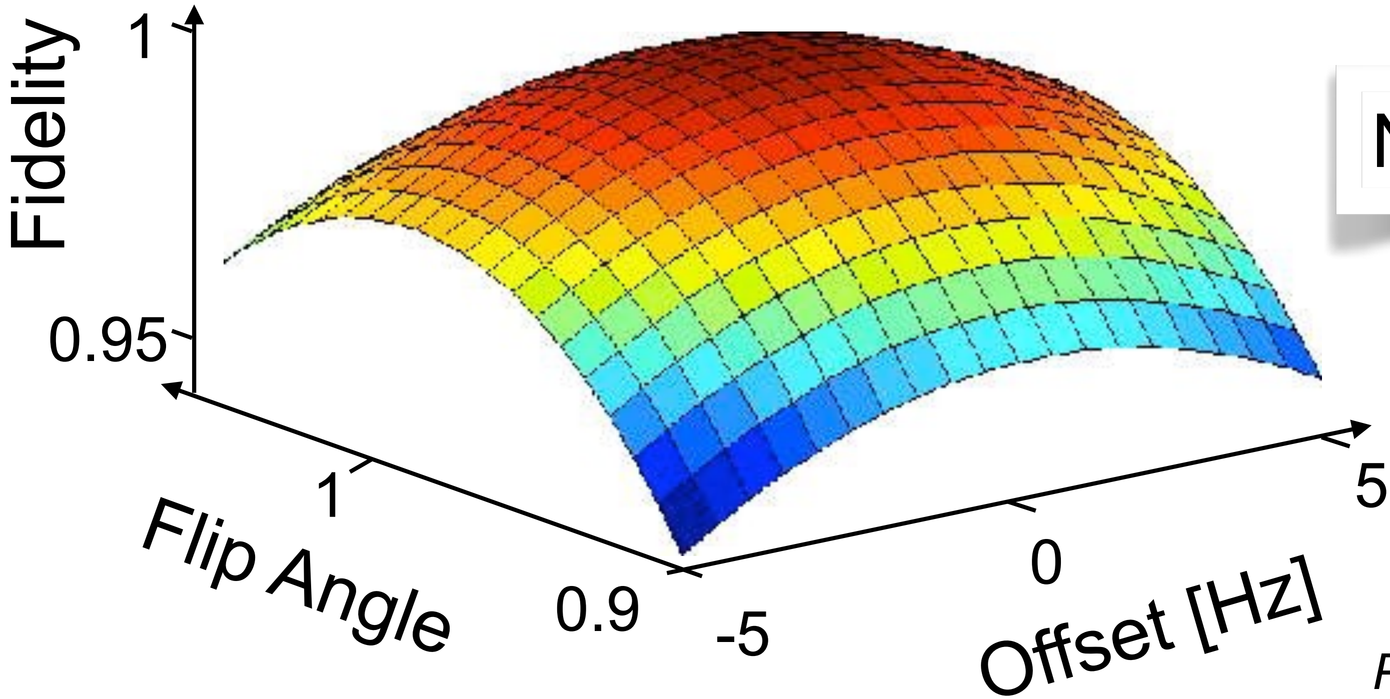
Design gate operations such that errors in experimental parameters cancel rather than amplify each other



Robust Optimal Control



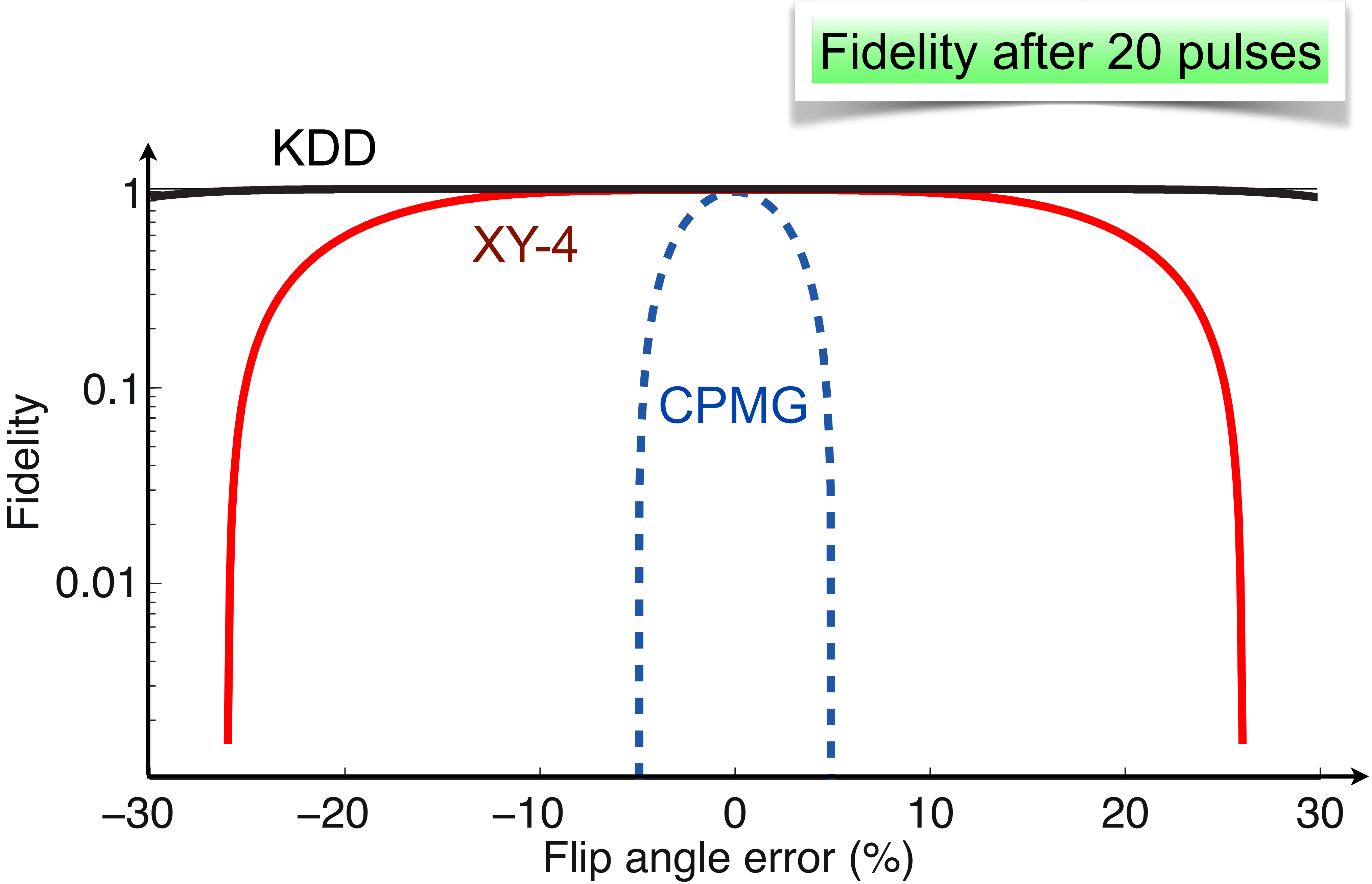




NMR example

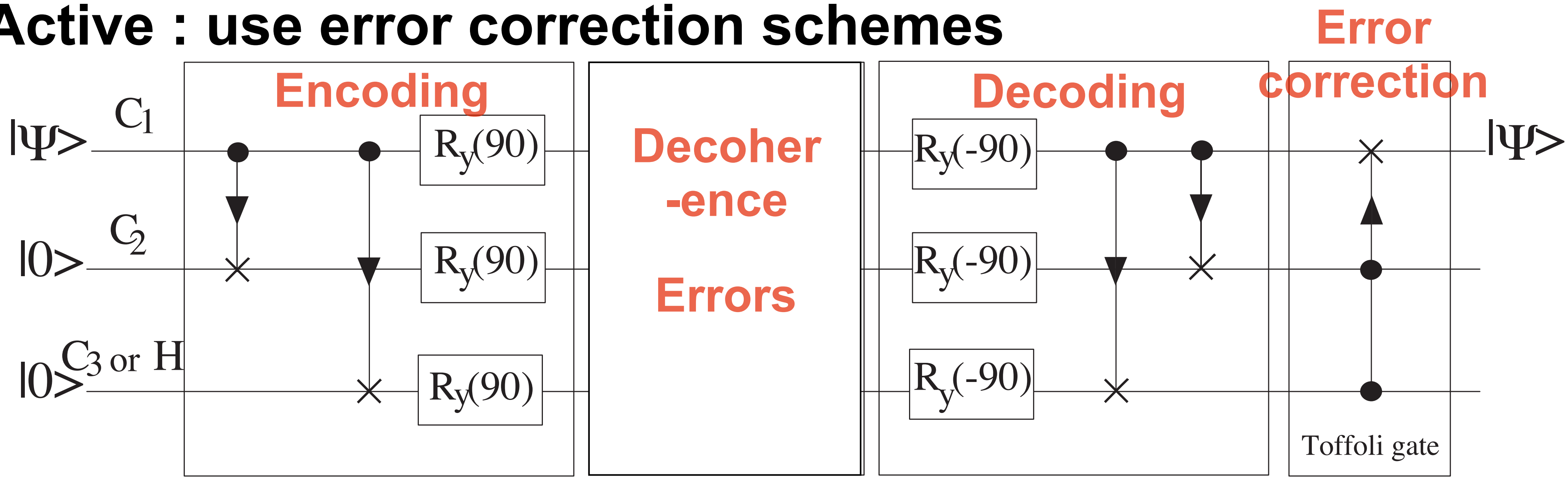
Phys. Rev. A 74, 062312 (2006)

Effect of Flip Angle Errors

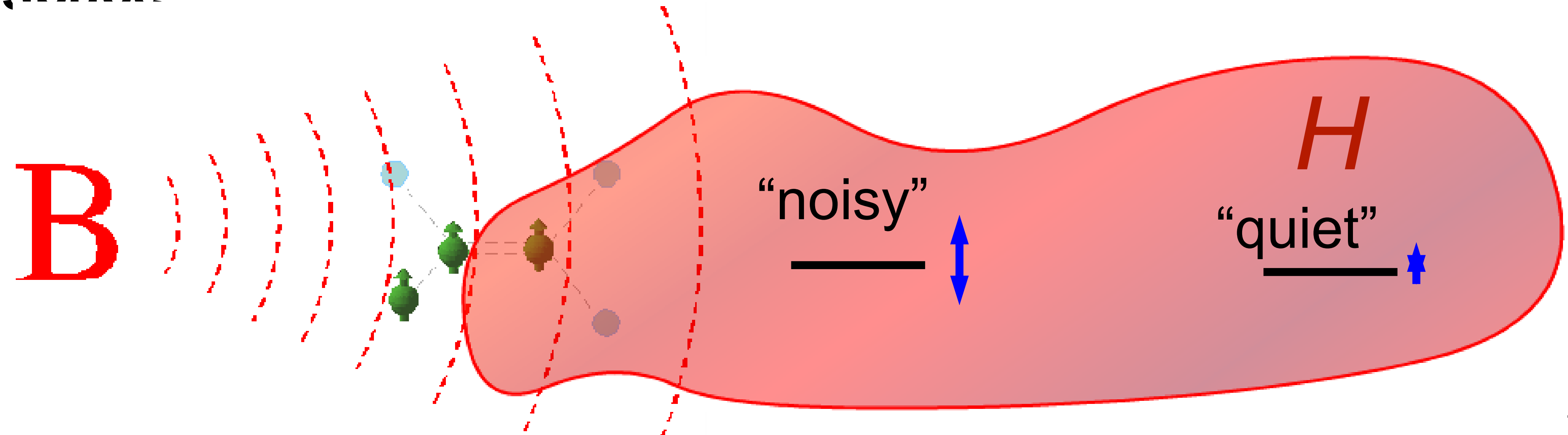


Active and Passive

Active : use error correction schemes



Passive : store information in "quiet" parts of Hilbert space



Types of Decoherence

Total decoherence: "anything goes"

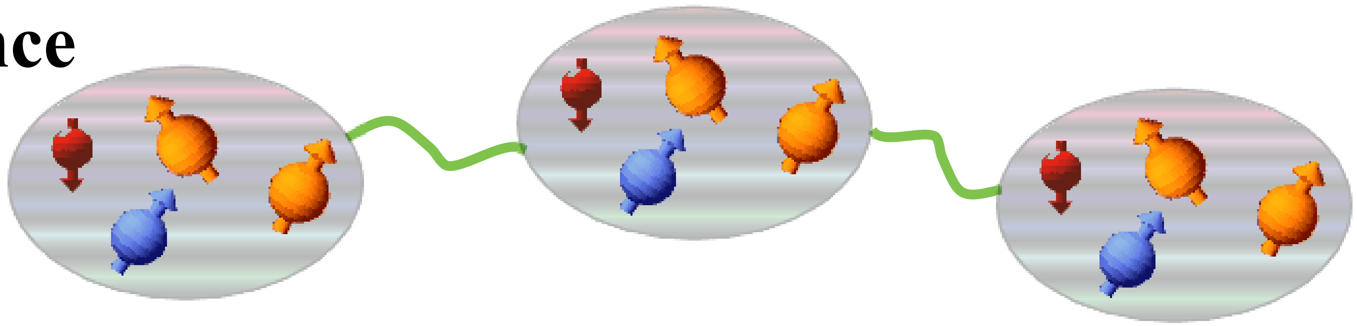
Independent qubit decoherence   Quantum Error Correction

Collective decoherence

coupling operator : $F_\alpha = \sum_i S^i_\alpha$: full permutation symmetry

Use symmetry to minimize decoherence

Cluster decoherence



Decoherence-free Subspaces

$$\mathcal{H} = \mathcal{H}_S \otimes \mathbf{1}_B + \mathbf{1}_S \otimes \mathcal{H}_B + \mathcal{H}_{\text{Int}}$$

System $\qquad$ Bath

$$\mathcal{H}_{\text{Int}} = \sum_\alpha \mathbf{F}_\alpha \otimes \mathbf{B}_\alpha$$

Equation of motion for system only:

$$\frac{d}{dt}\boldsymbol{\rho}_S = -\frac{i}{\hbar}[\tilde{\mathcal{H}}_S, \boldsymbol{\rho}_S] + \boxed{\frac{1}{2}\sum_{\alpha,\beta} a_{\alpha\beta}\left([\mathbf{F}_\alpha, \boldsymbol{\rho}_S \mathbf{F}_\beta^\dagger] + [\mathbf{F}_\alpha \boldsymbol{\rho}_S, \mathbf{F}_\beta^\dagger]\right)}.$$

decoherence

Decoherence-free subspace : states |i>

Relevant density operator : $\tilde{\boldsymbol{\rho}} = \sum_{i,j} \tilde{\rho}_{i,j} |i\rangle\langle j|$

Decoherence-free subspace exists if

$$\sum_{\alpha,\beta} a_{\alpha\beta}\left([\mathbf{F}_\alpha, \tilde{\boldsymbol{\rho}} \mathbf{F}_\beta^\dagger] + [\mathbf{F}_\alpha \tilde{\boldsymbol{\rho}}, \mathbf{F}_\beta^\dagger]\right) = 0$$

Useful case: $\qquad$ (i.e. |i>are degenerate set of eigenstates)

$$\mathbf{F}_\alpha |i\rangle = c_\alpha |i\rangle$$

Capacity of DFS

How much information can be stored in DFS?

Collective decoherence: $F_\alpha = \sum_i S_\alpha^i \; ; \;\; \alpha = x, y, z$

DFS spanned by singlet states

states M_z=0

M_z=0

S > 0

$$\dim[\mathrm{DFS}(K)] = \binom{K}{K/2} - \binom{K}{K/2-1} = \frac{K!}{(K/2)!\,(K/2+1)!}$$

\# of qubits $N = \log_2 \dim[\mathrm{DFS}(K)] = K - \frac{3}{2}\log_2 K + \mathcal{O}(1)$

$\Rightarrow$ **DFS asymptotically fill complete Hilbert space**

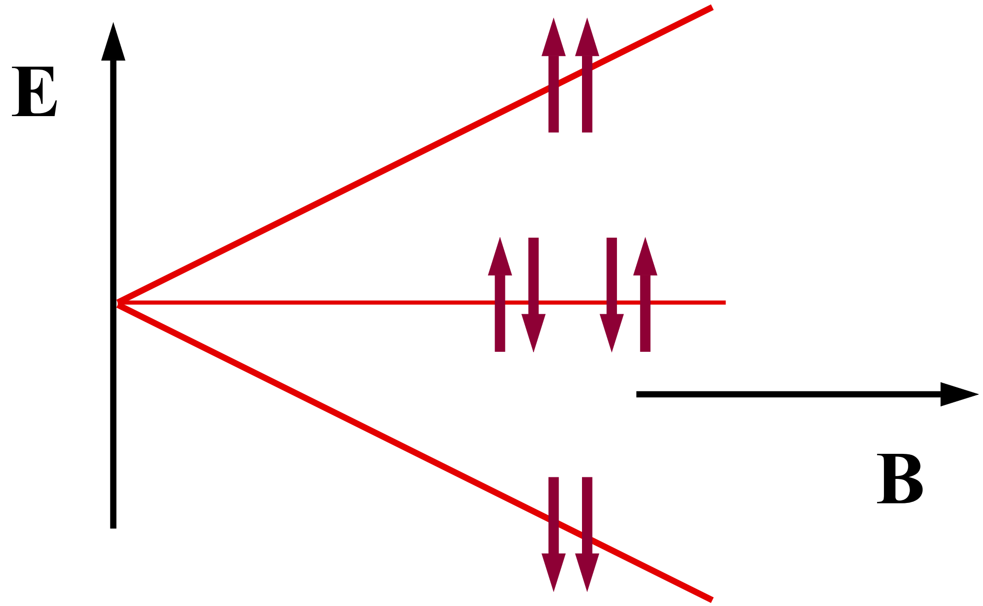
$$\left.\begin{array}{l}\text{Total decoherence}\\ \text{Individual qubit decoherence}\end{array}\right\} \quad \dim[\mathrm{DFS}(K)] \rightarrow 0$$

DFS for Spins

Perturbation : $H_z = b(t) \sum_i I_z^i$

Dephasing : $i\hbar \frac{d}{dt} \rho_{ij} = b(t)\, \Delta m_{ij} \rho_{ij}$ $\qquad \Delta m_{ij} = \langle i|\sum_k I_z^k|i\rangle - \langle j|\sum_k I_z^k|j\rangle$

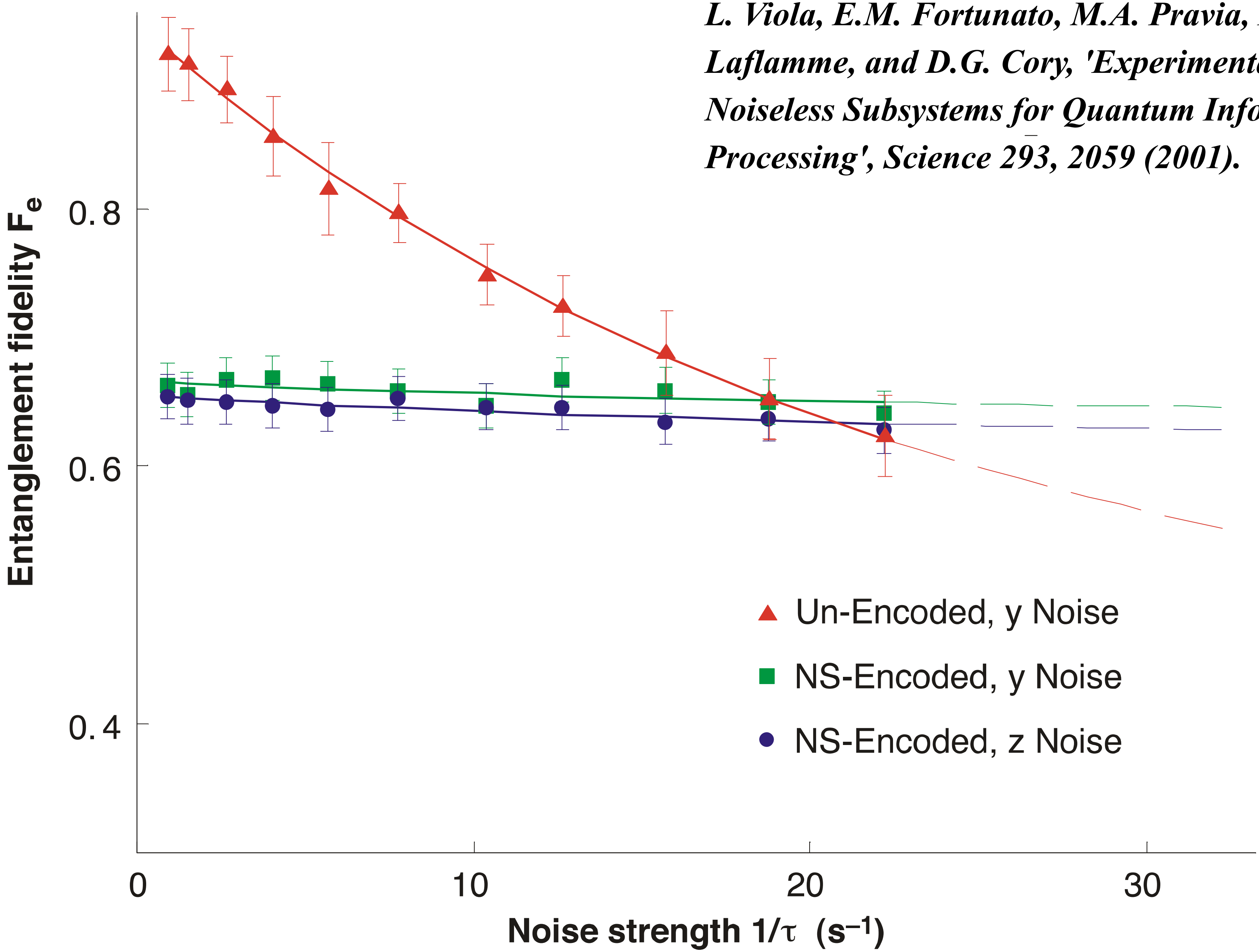
Example 2 spins 1/2



A subspace of this is also immune to noise || x

Implementation

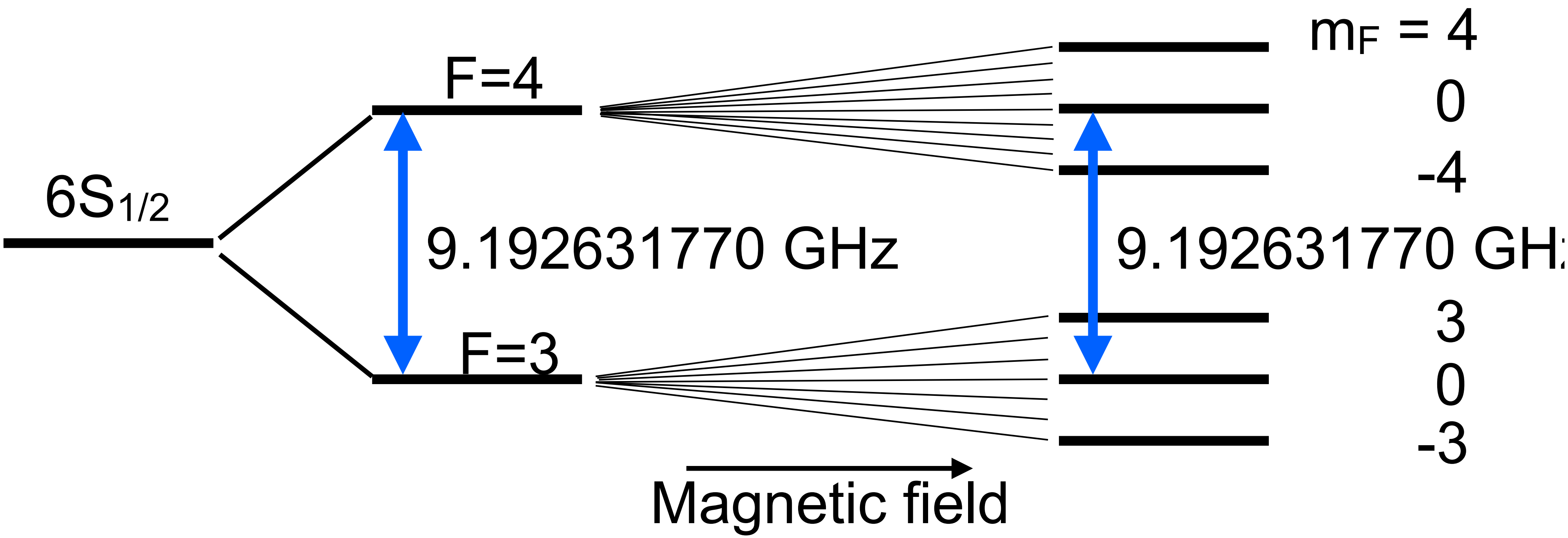
L. Viola, E.M. Fortunato, M.A. Pravia, E. Knill, R. Laflamme, and D.G. Cory, 'Experimental Realization of Noiseless Subsystems for Quantum Information Processing', Science 293, 2059 (2001).



Clock Transitions

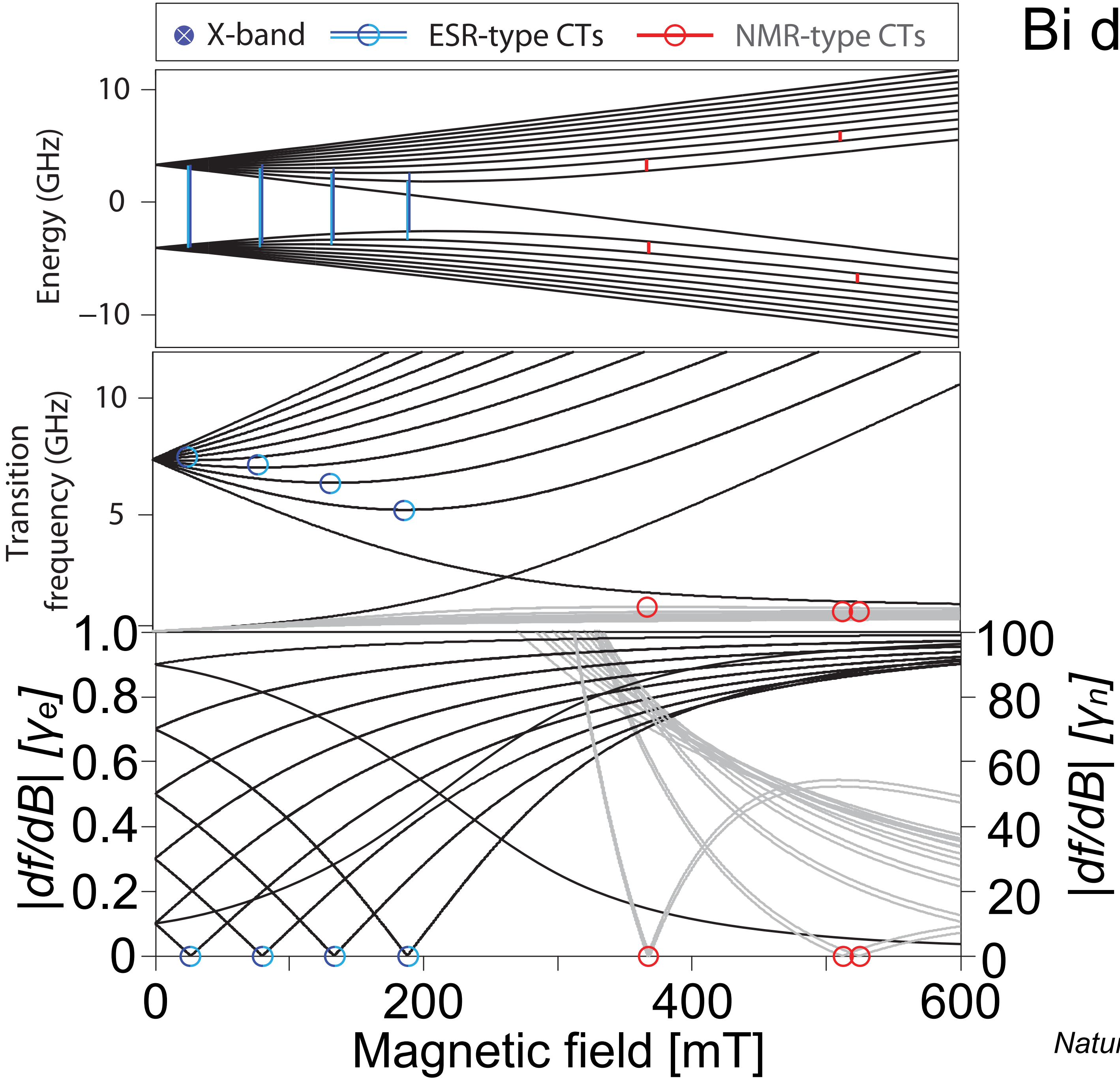
Definition

1 second = 9192631770 periods of the
^{133}Cs, F = 3, m_F = 0 ↔ F = 4, m_F = 0 hyperfine transition



Clock Transition for Qubit

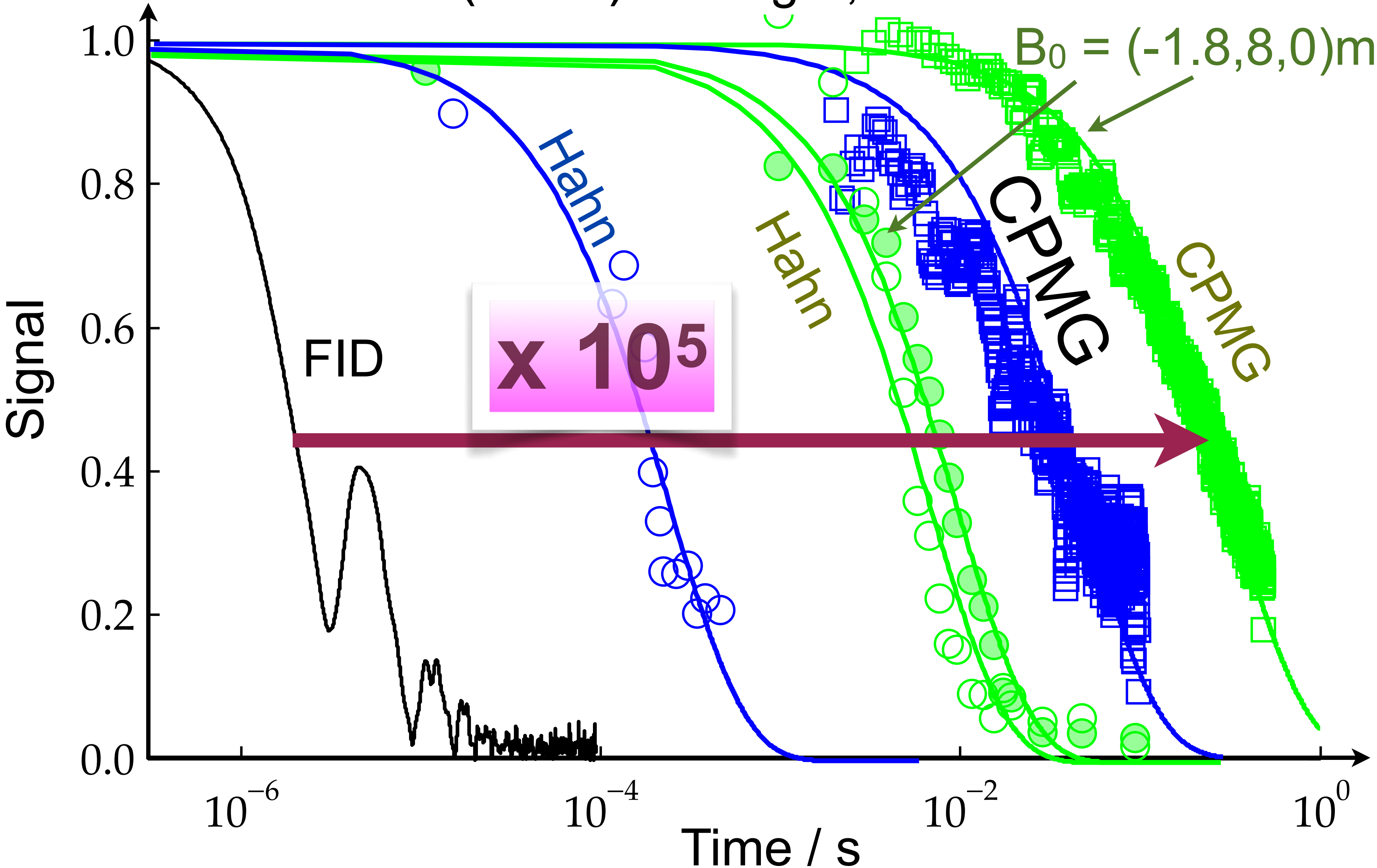
Bi donor in Si



Nature Nano. 8, 561 (2013).

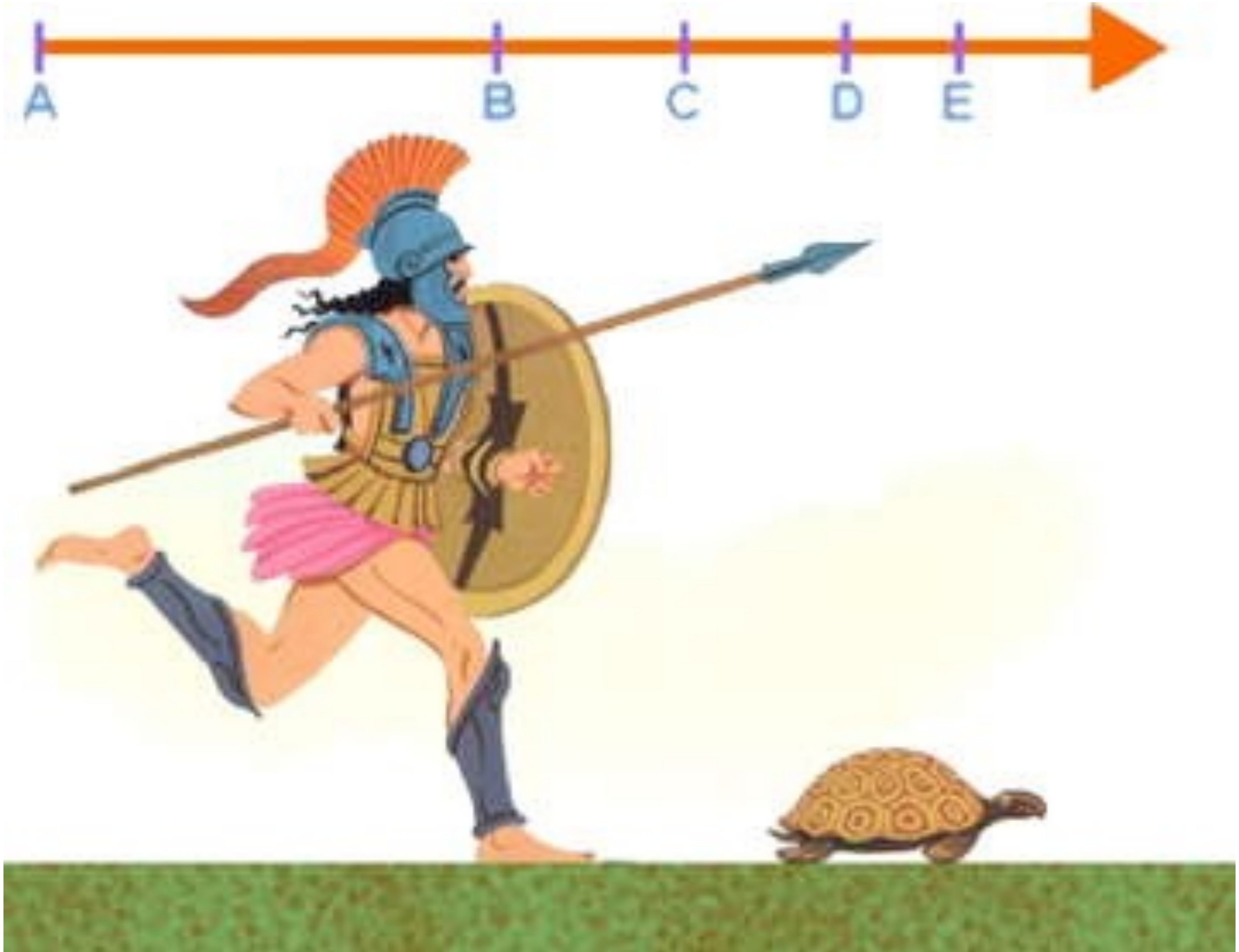
Keeping a Photon Alive

store it in $Pr^{3+}:La_2(WO_4)_3$ + longer; current record : 6 hours



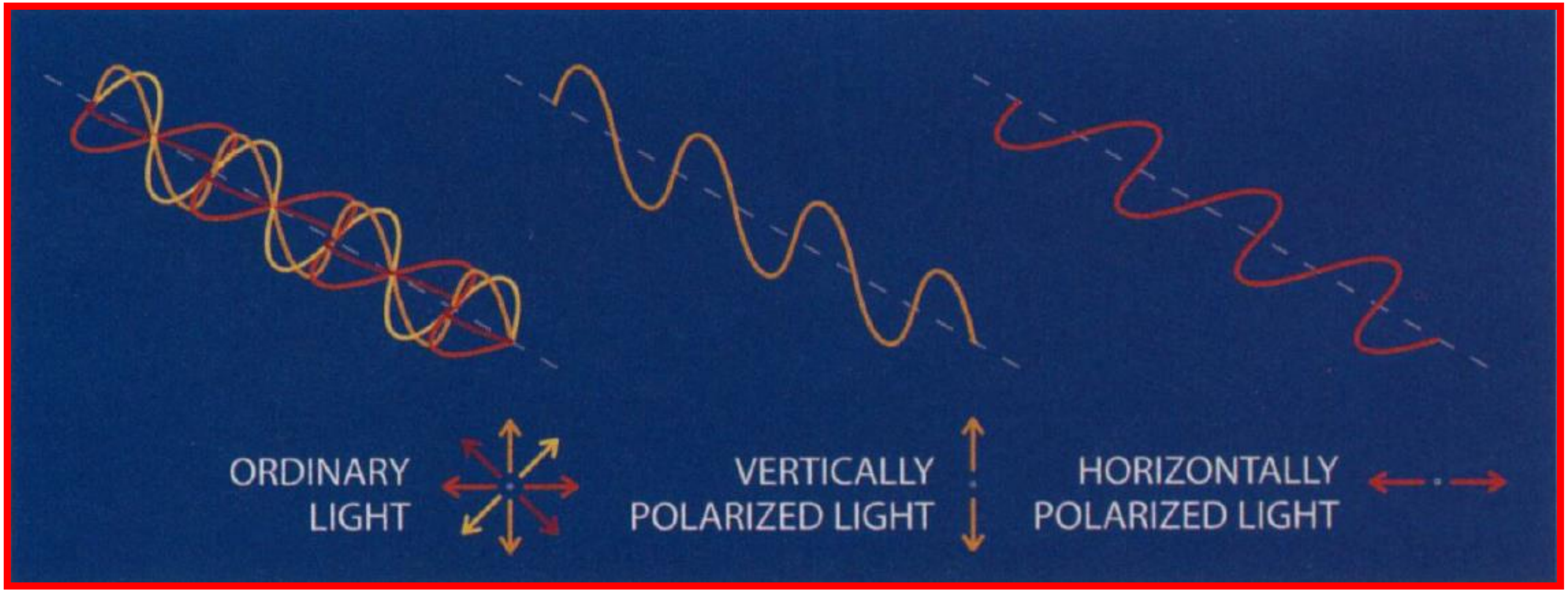
Zeno Effect

Zeno of Elea (ca. 490-430 b.c.)



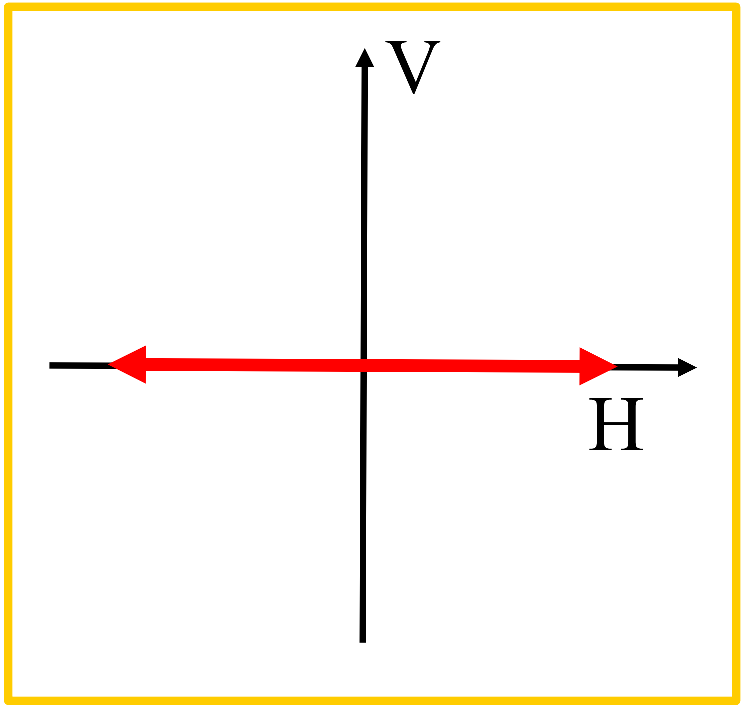
Two level system (TLS)

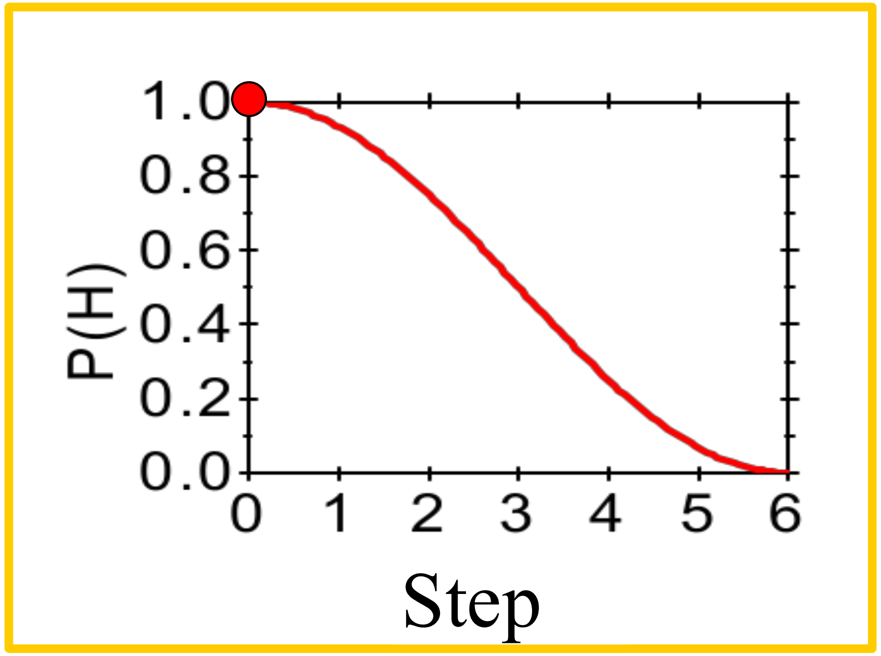
Polarization of light



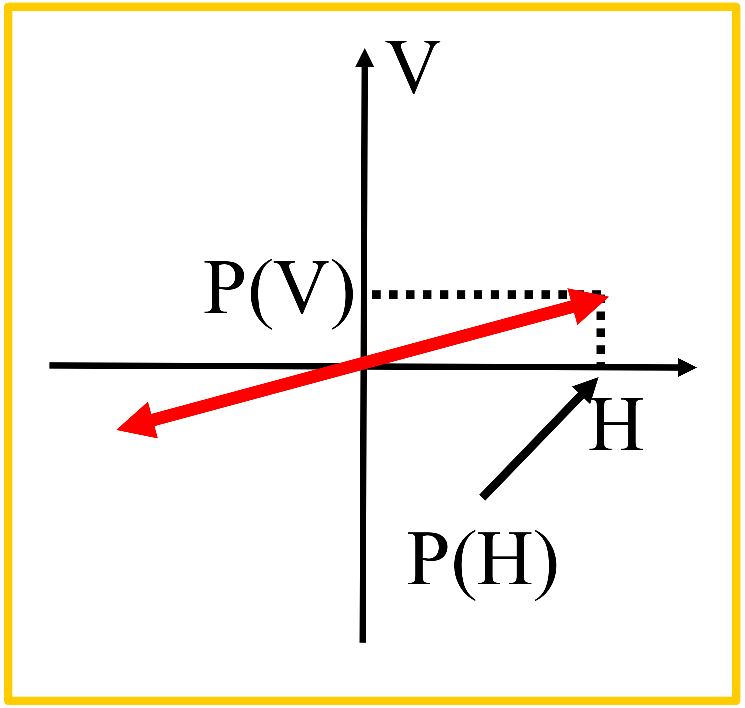
Kwiat, Winfurter, Zeilinger. Sci. Am. November 52 (1996).

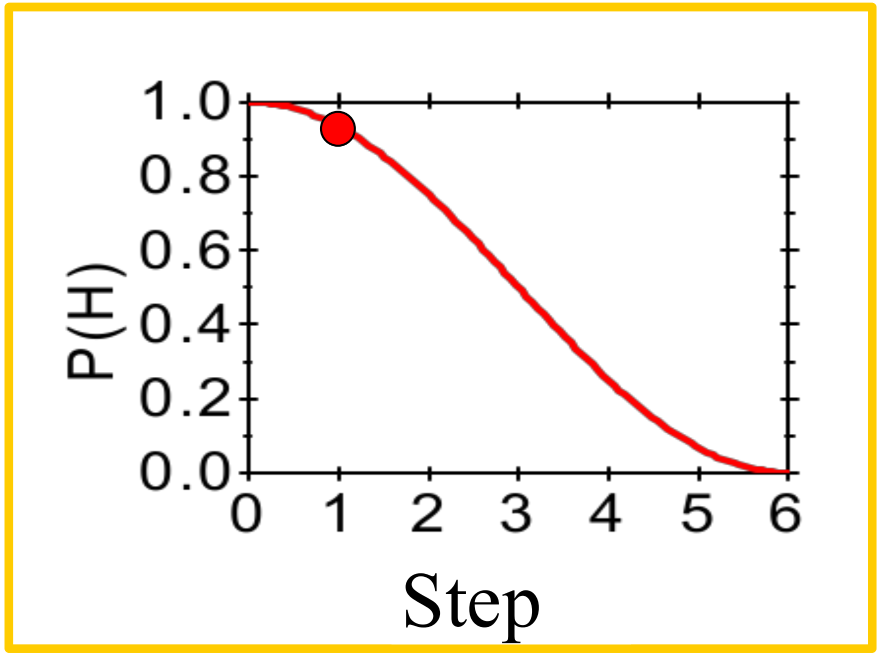
TLS Dynamics



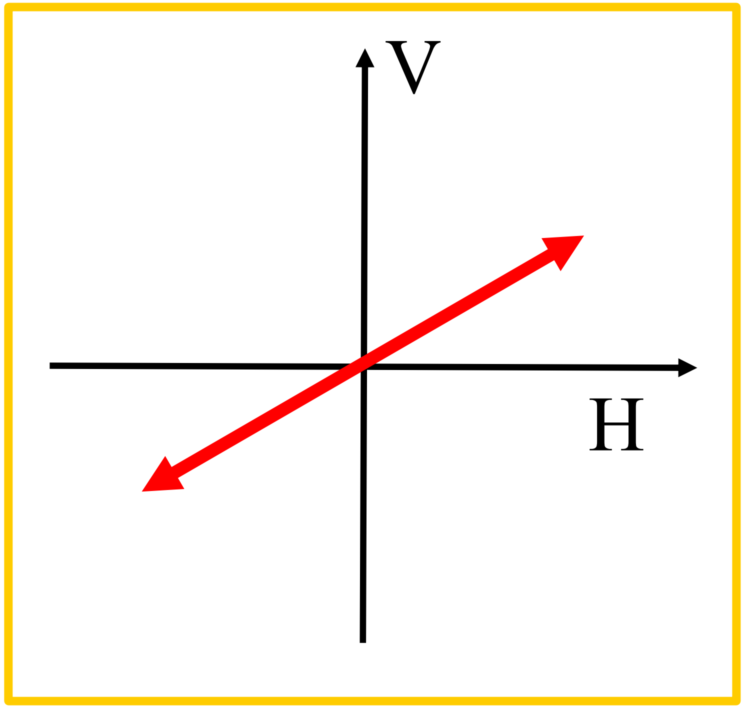


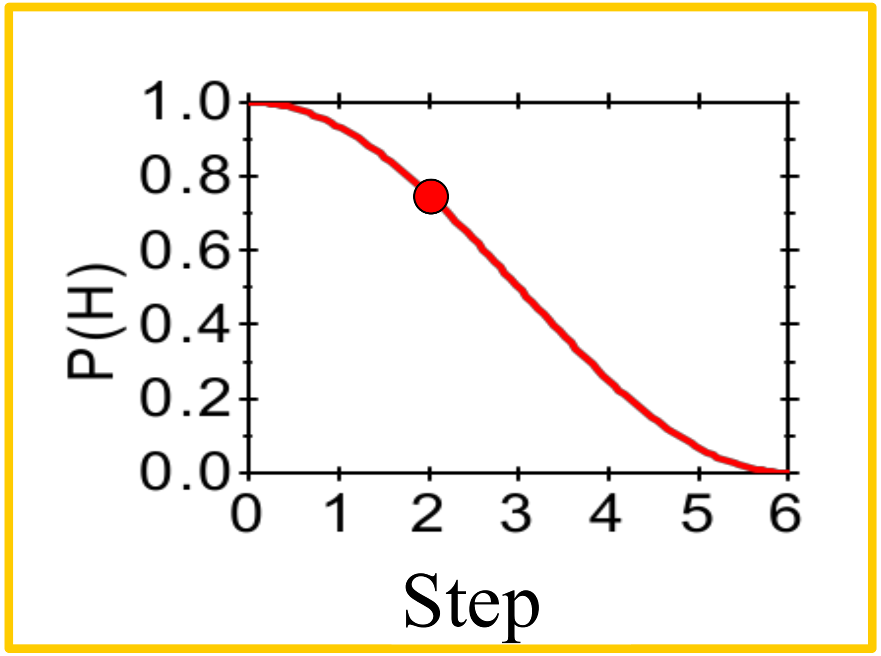
TLS Dynamics



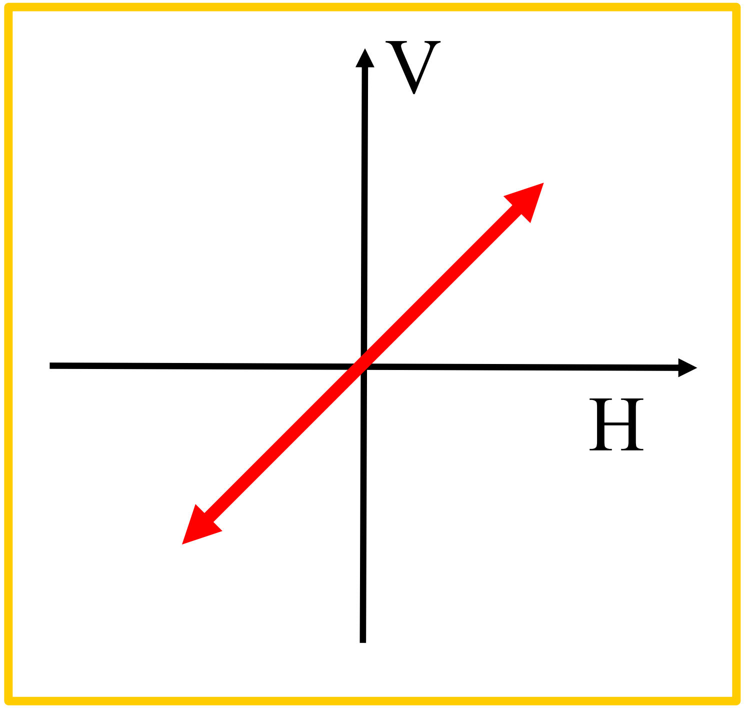


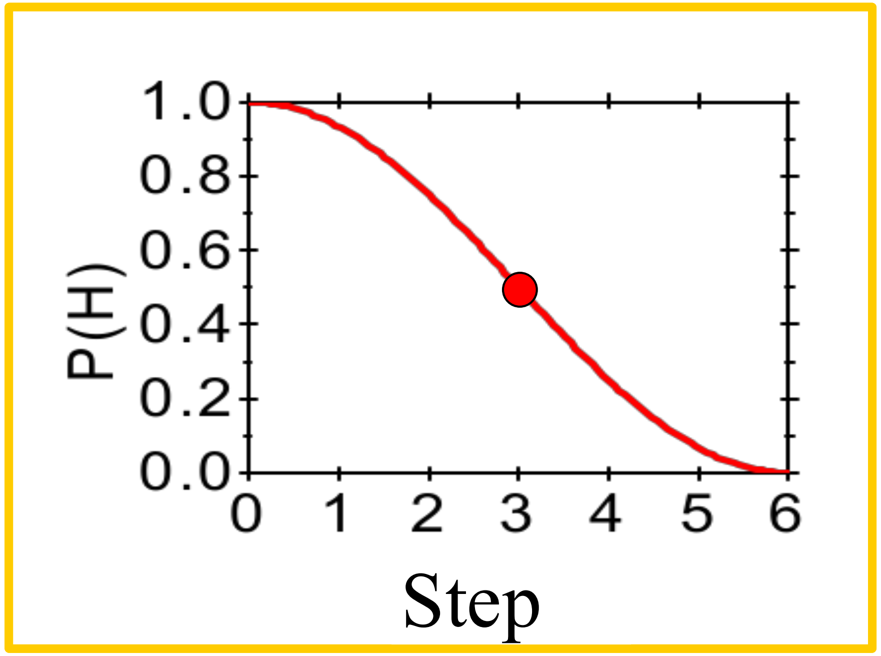
TLS Dynamics



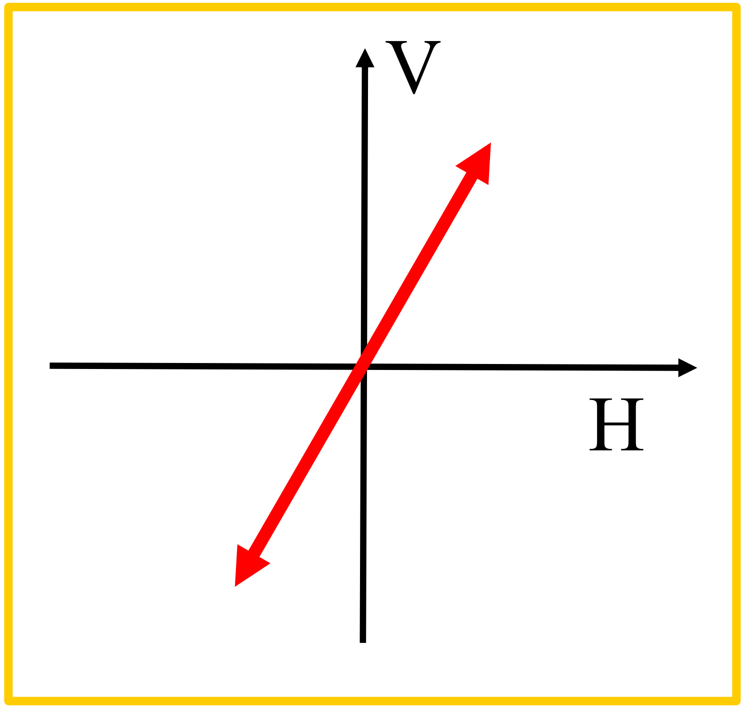


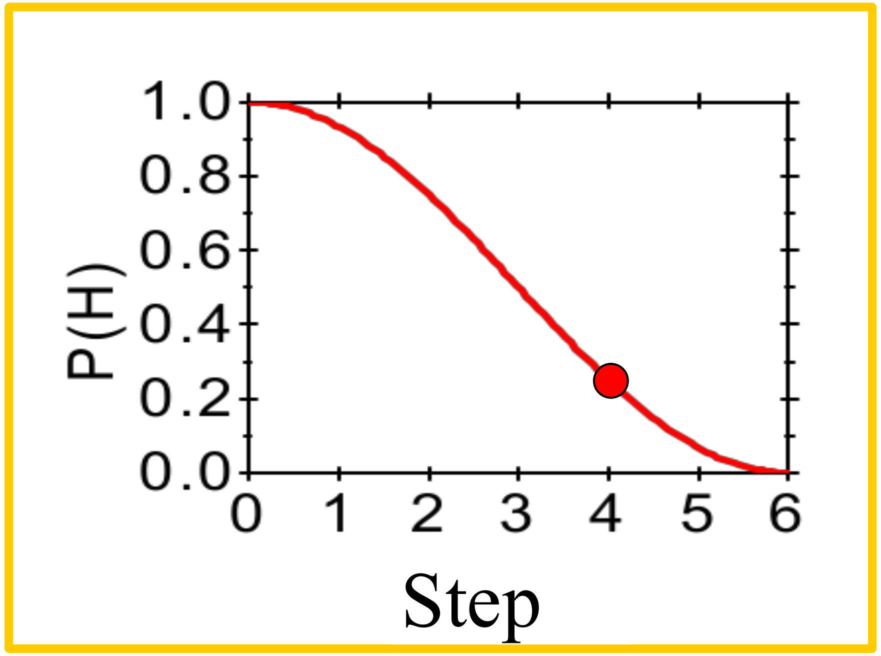
TLS Dynamics



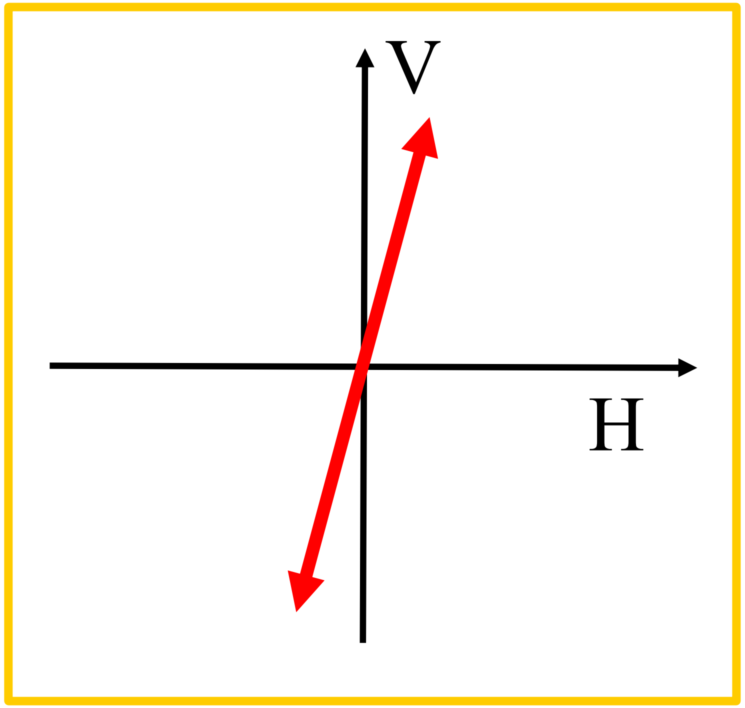


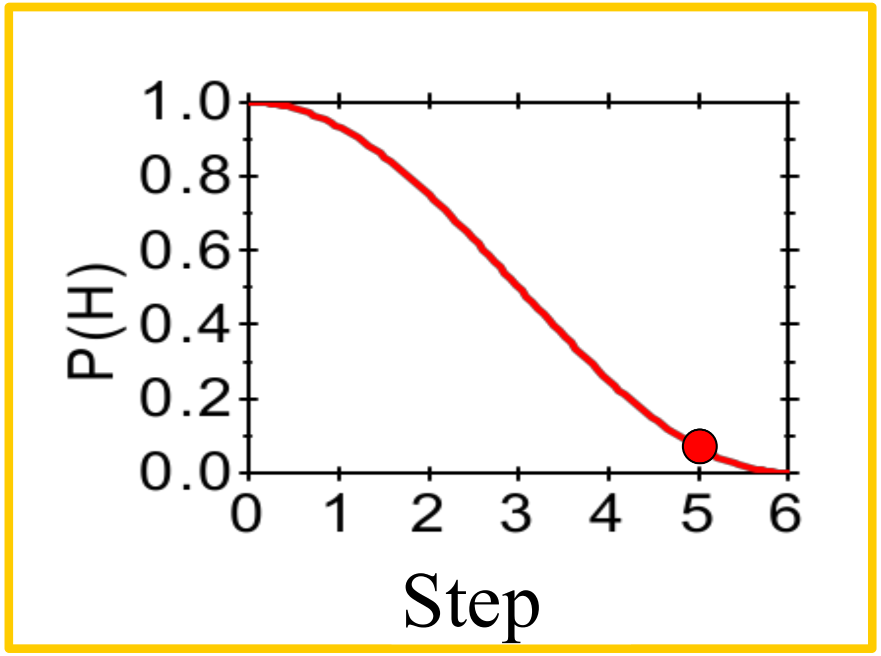
TLS Dynamics



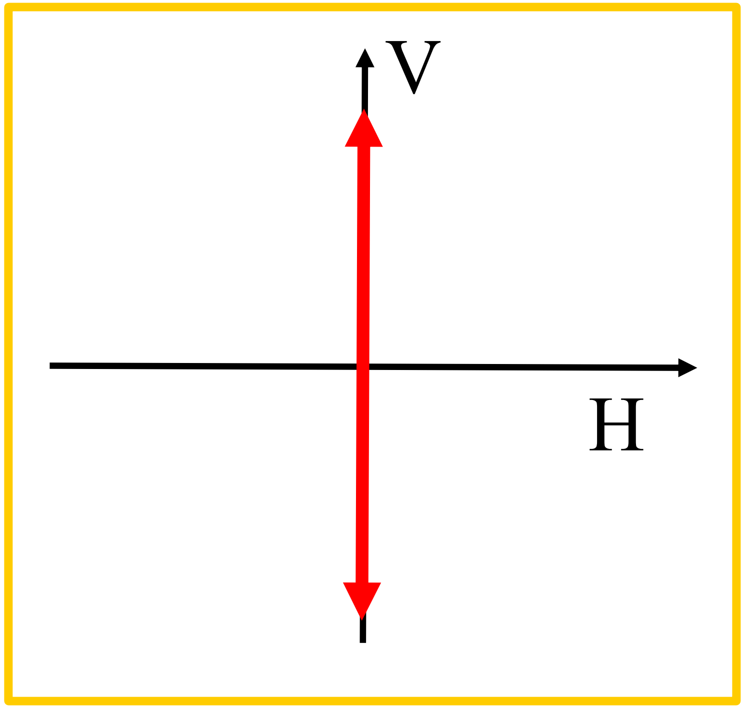


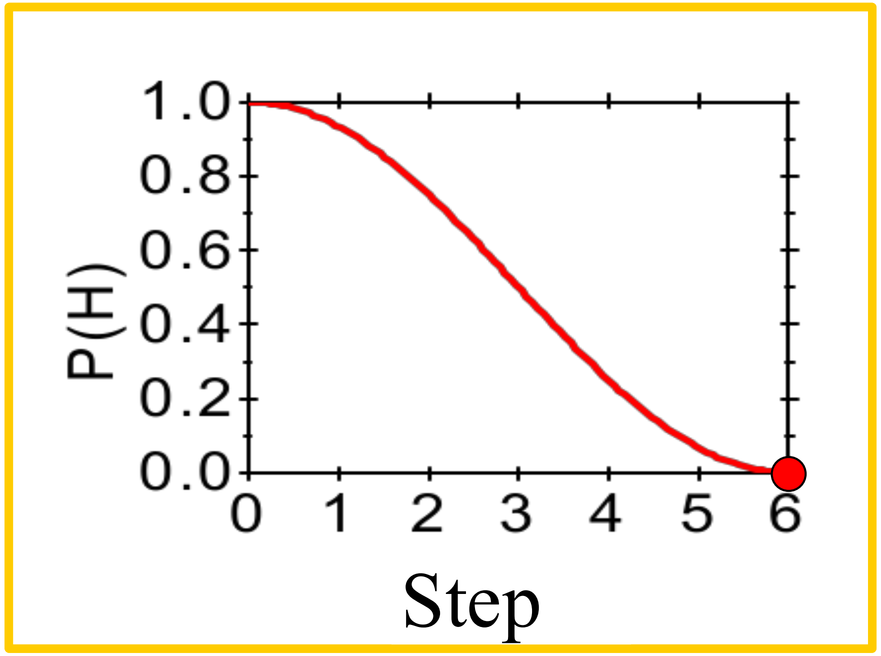
TLS Dynamics



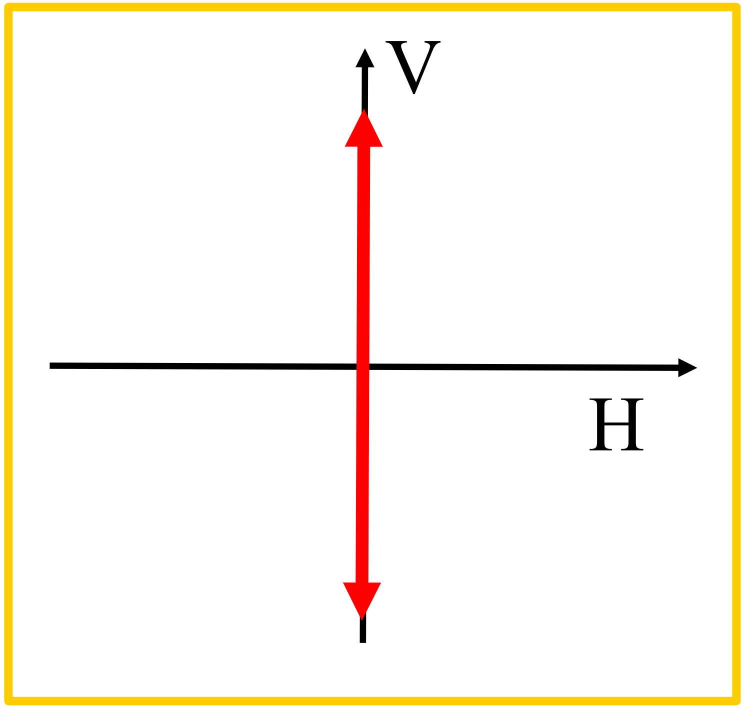


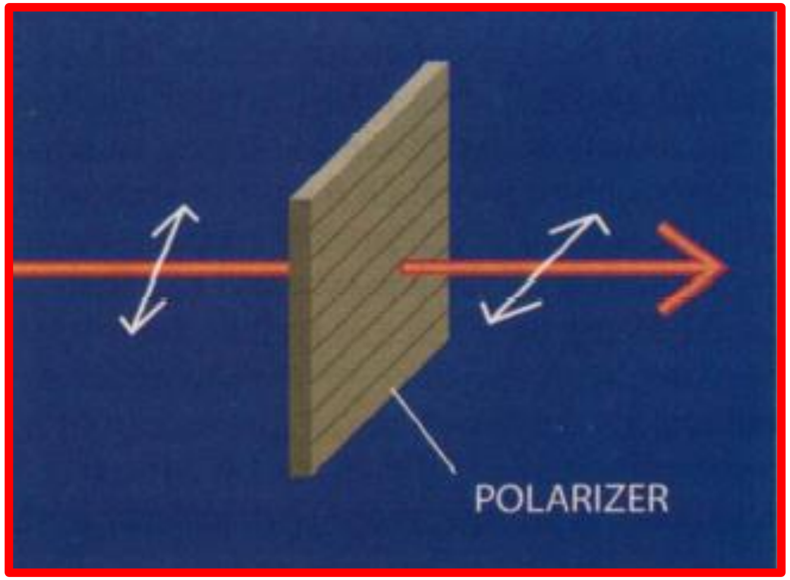
TLS Dynamics



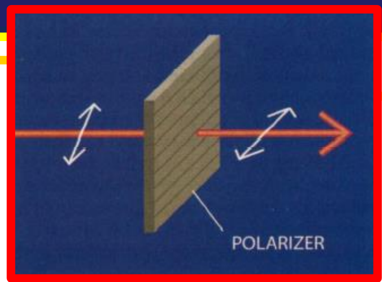


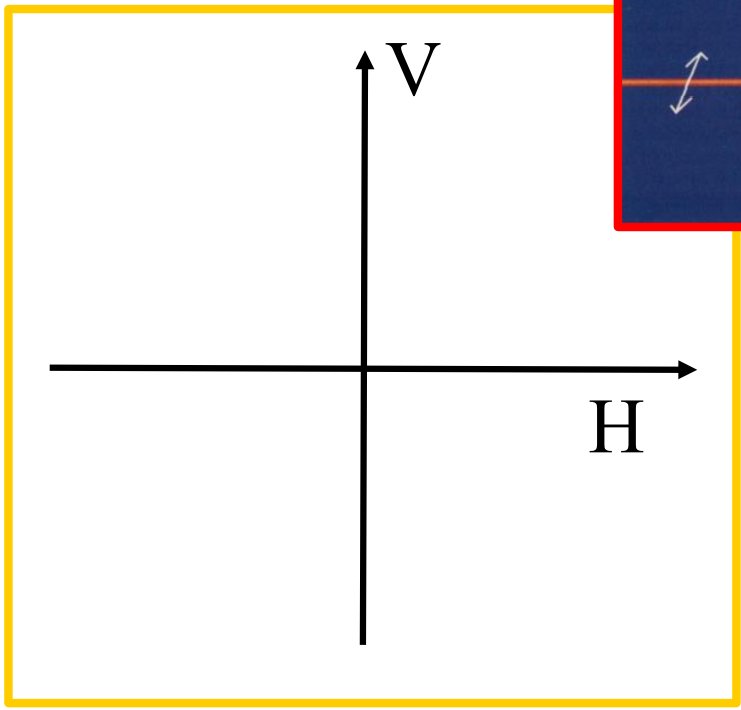
TLS Dynamics

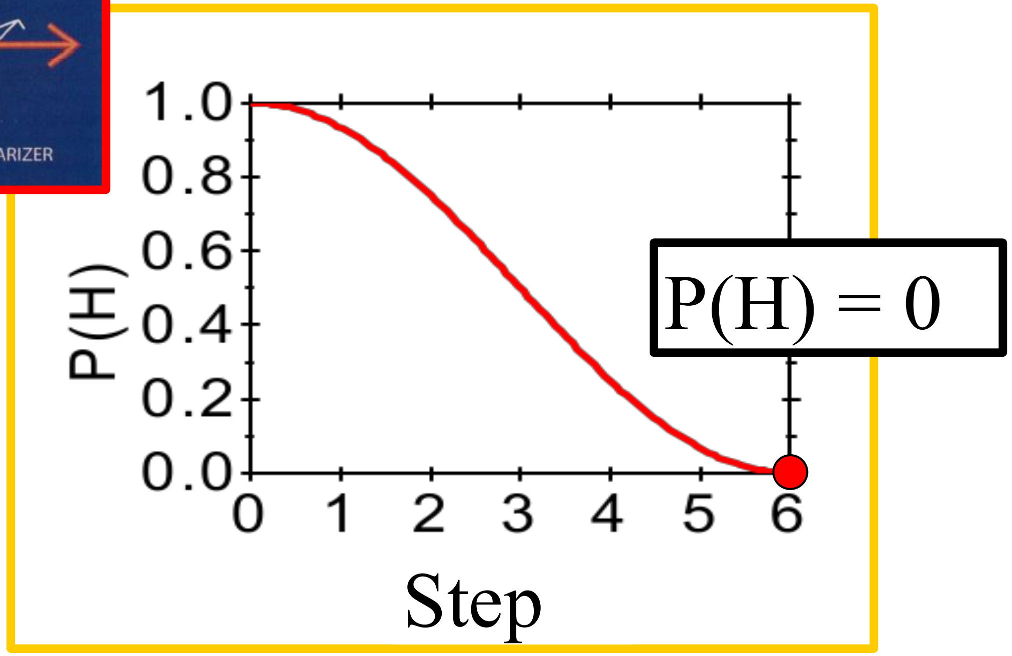




TLS Dynamics

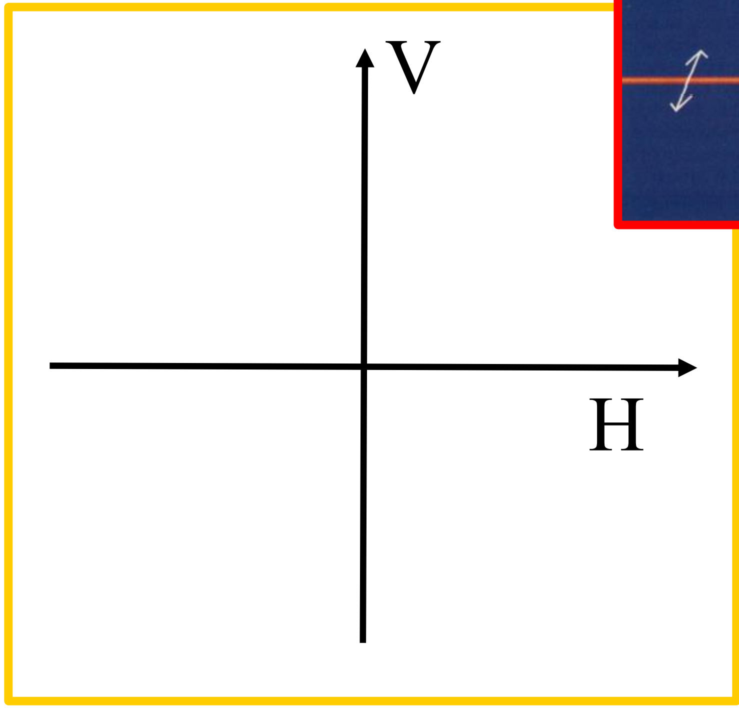


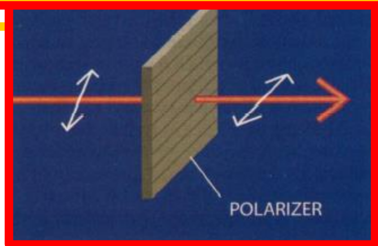


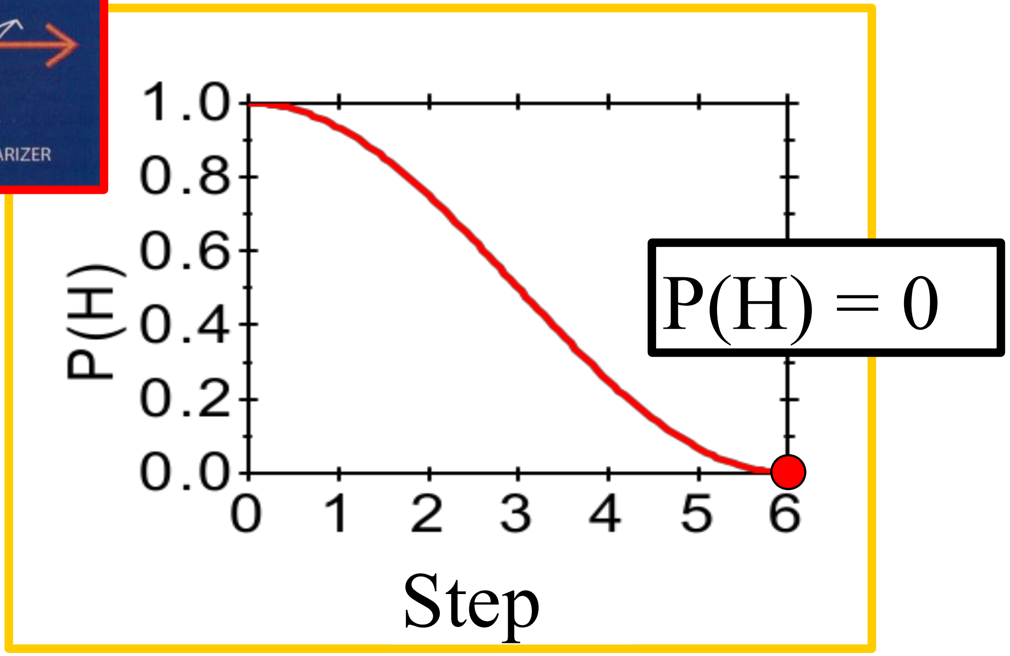


TLS Dynamics

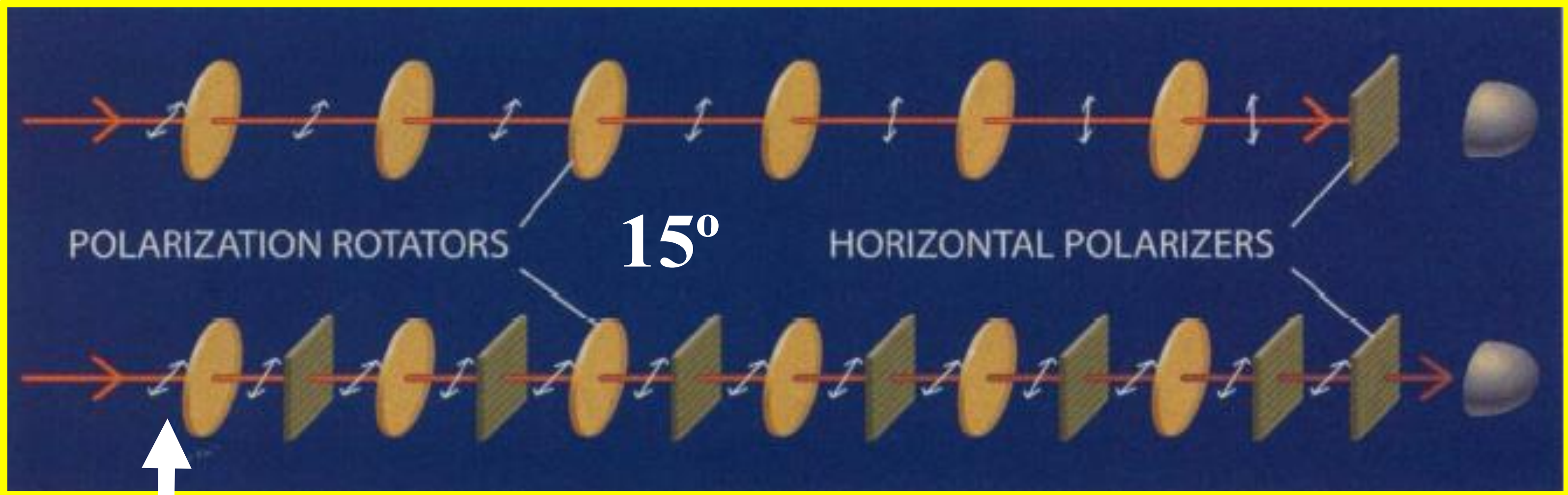
We can avoid the rotation evolution through measurements

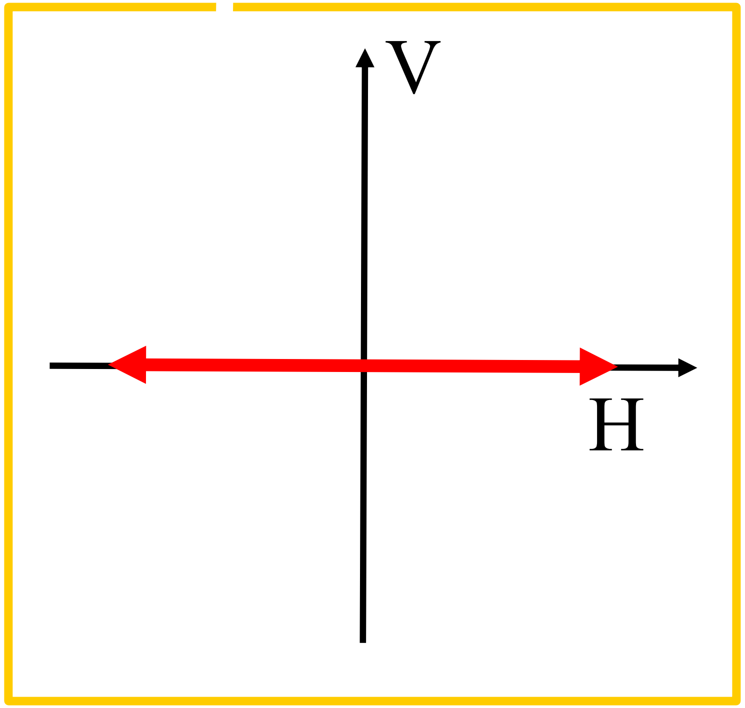


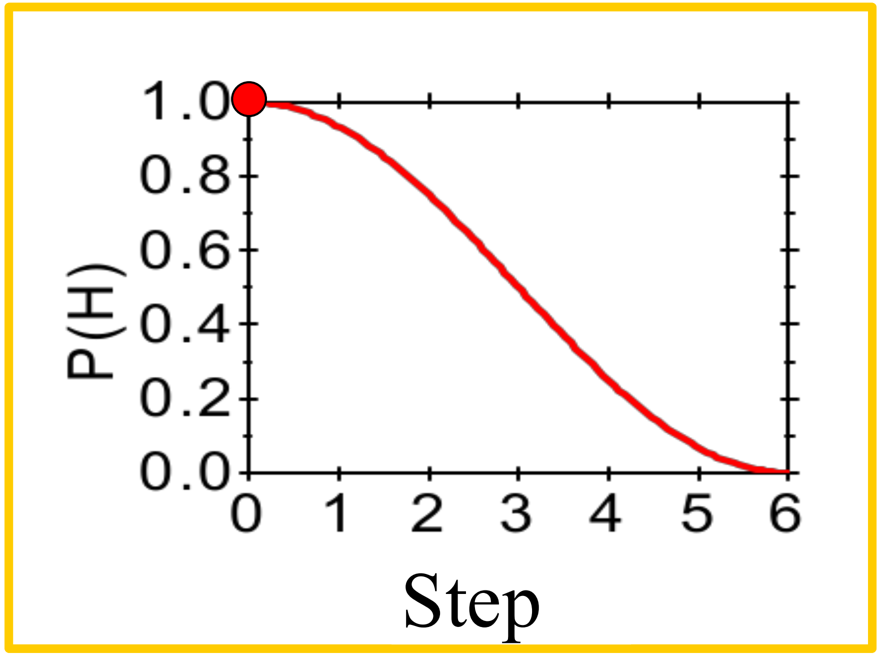




Stroboscopically measured TLS Dynamics







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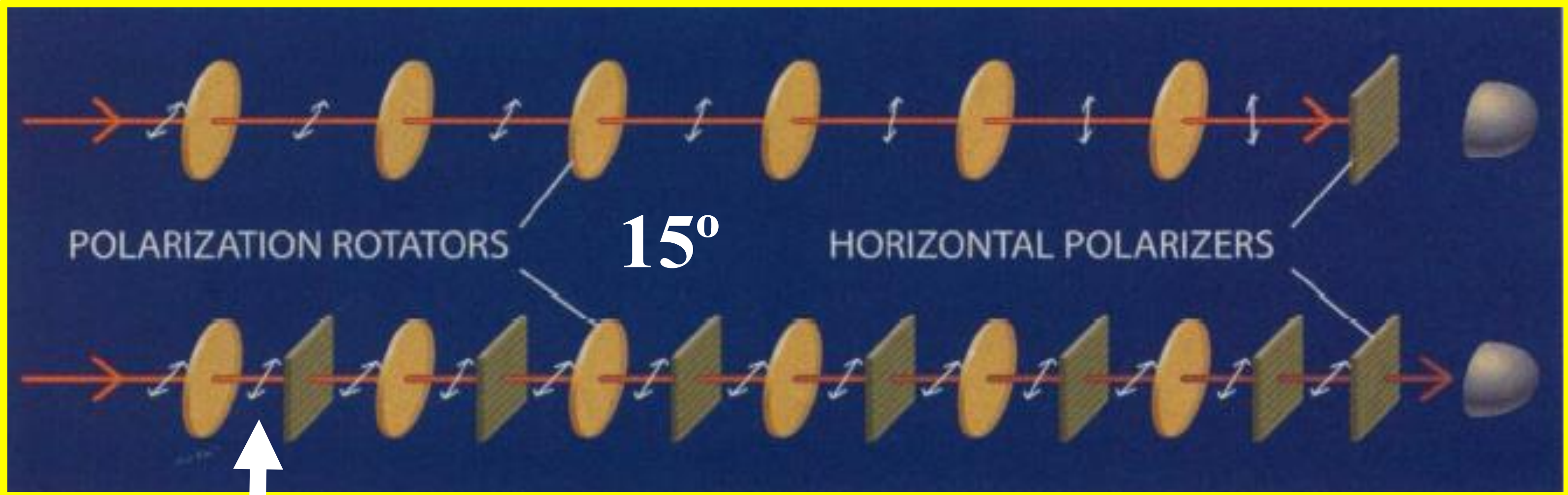

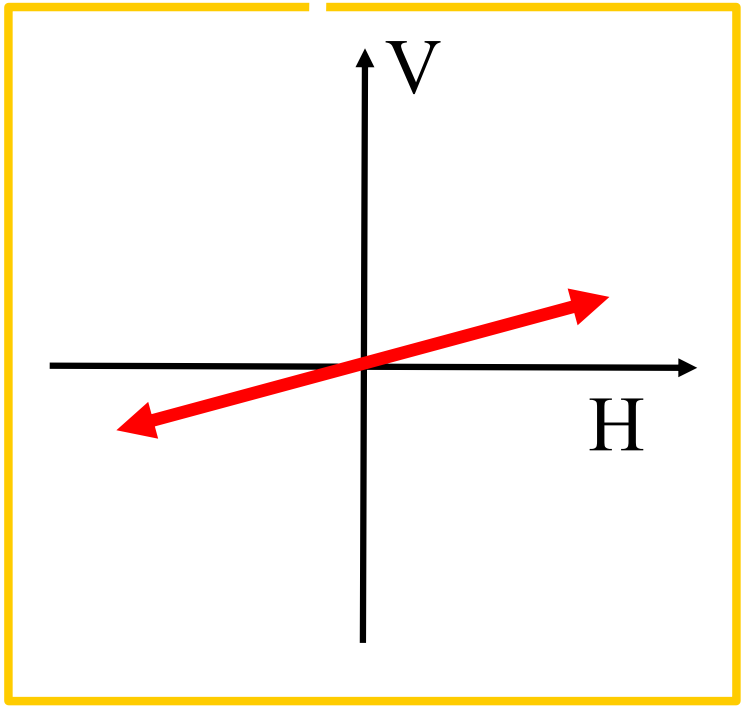

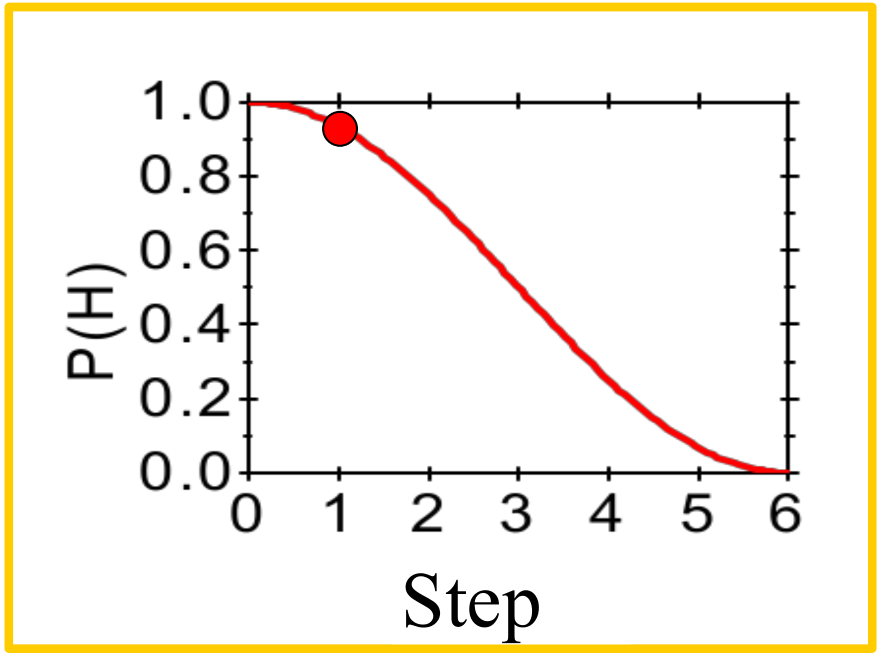

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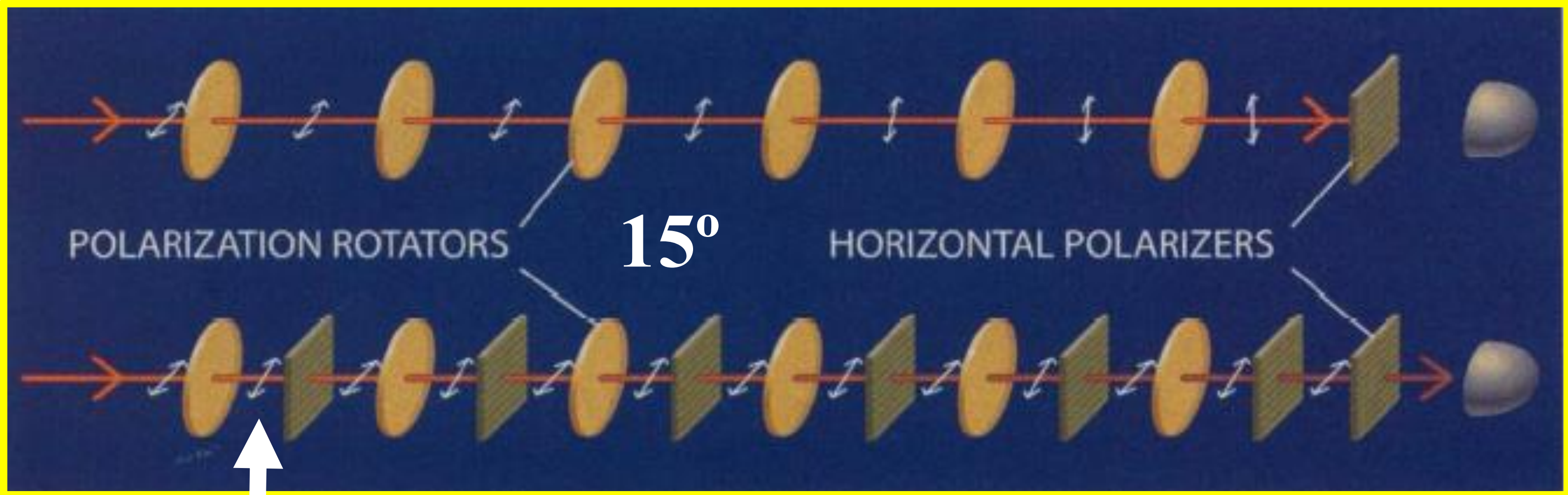

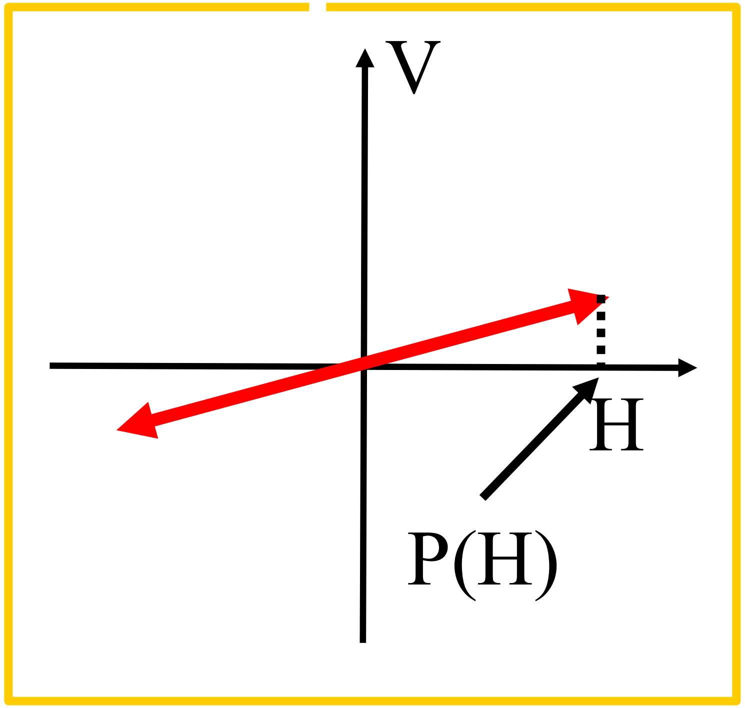

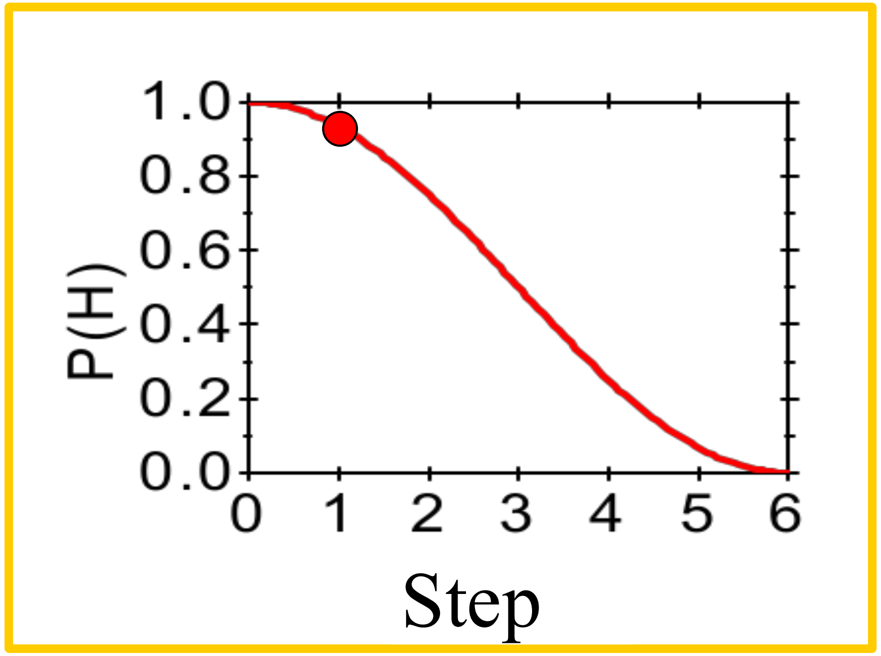

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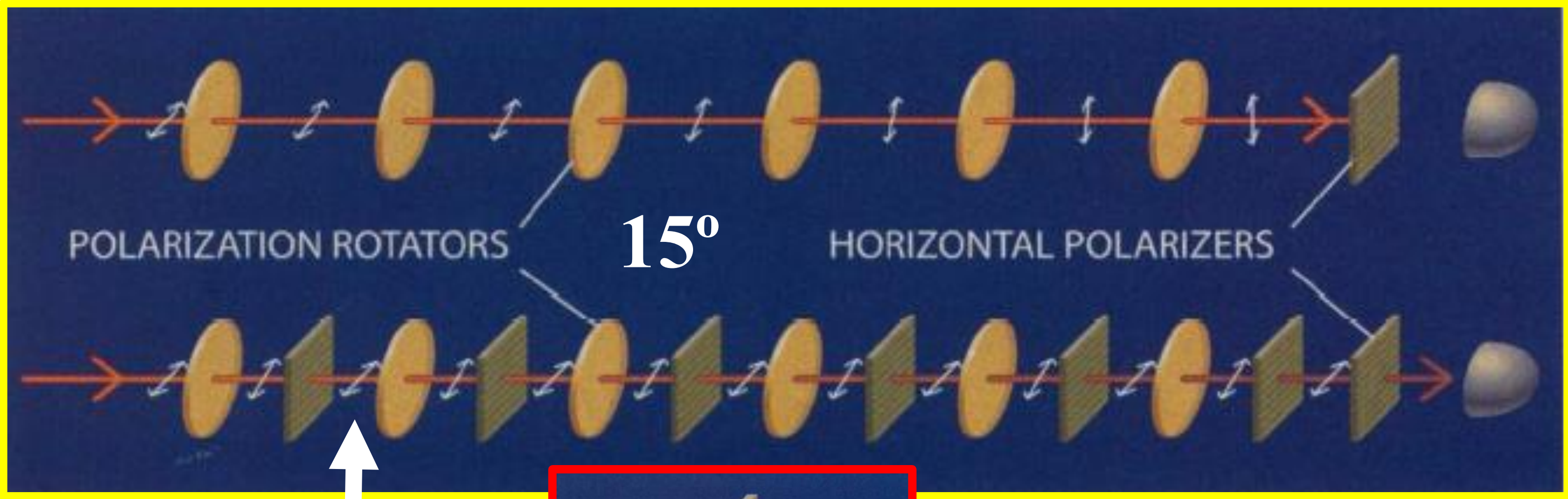

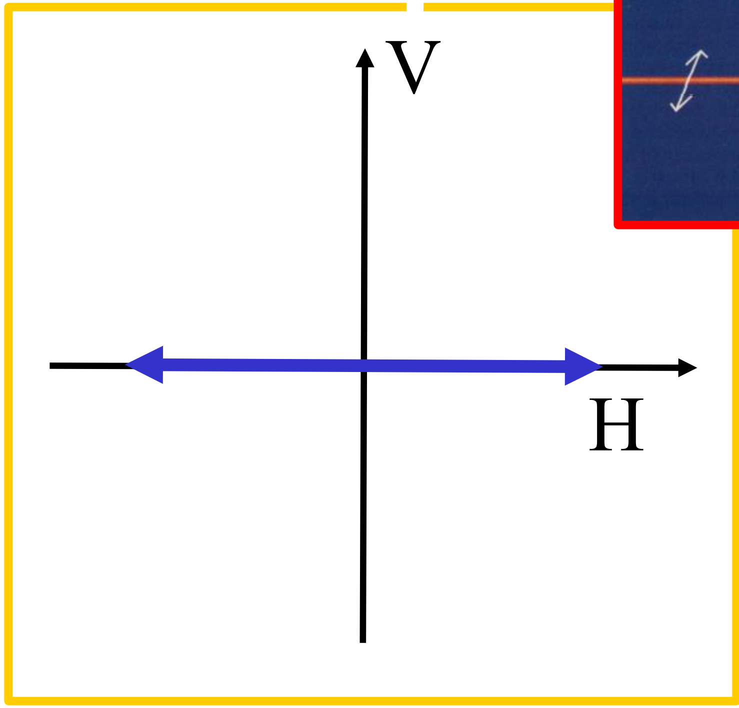

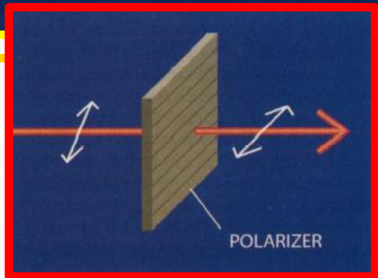

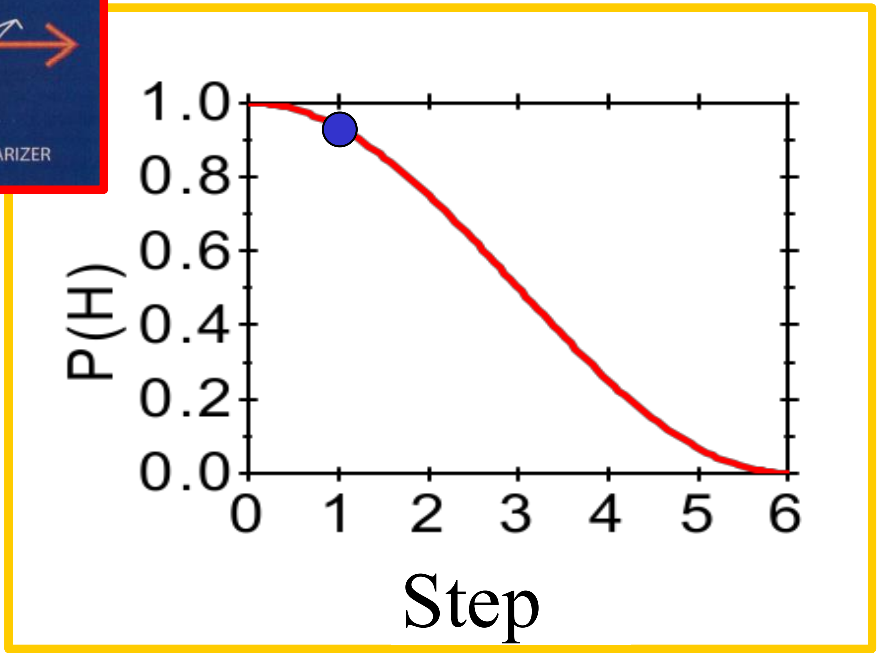

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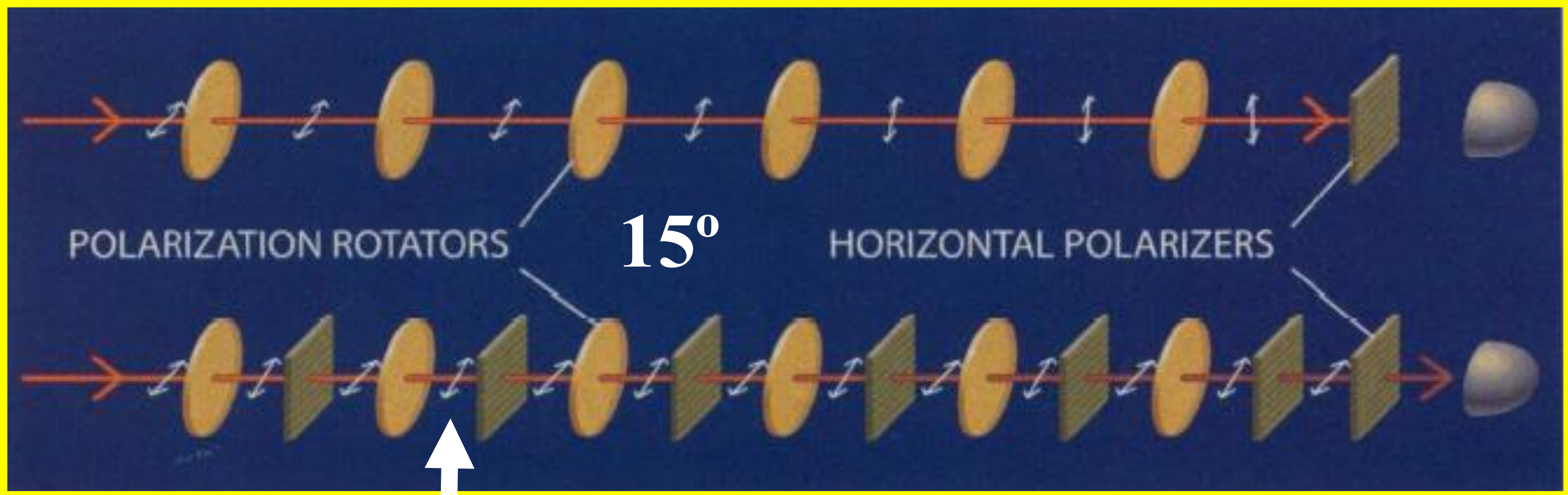

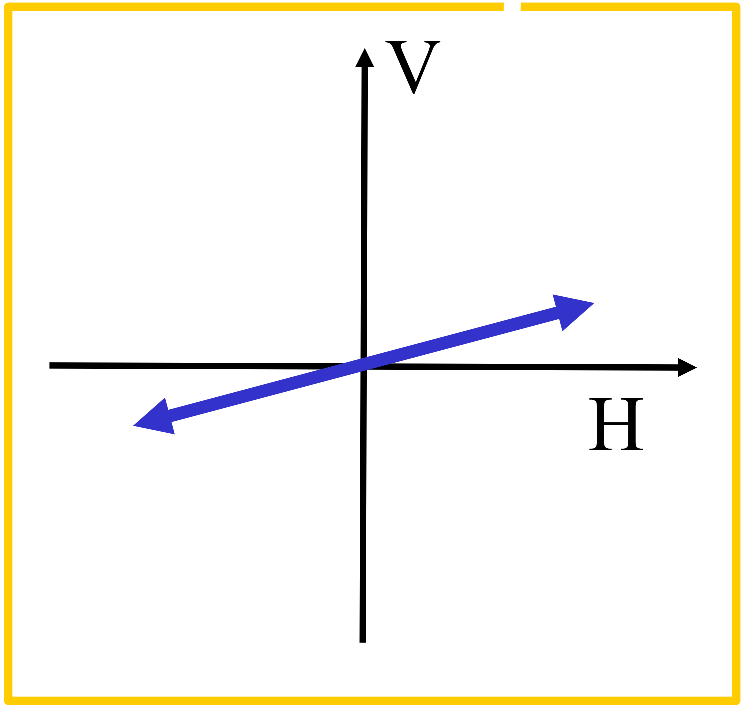

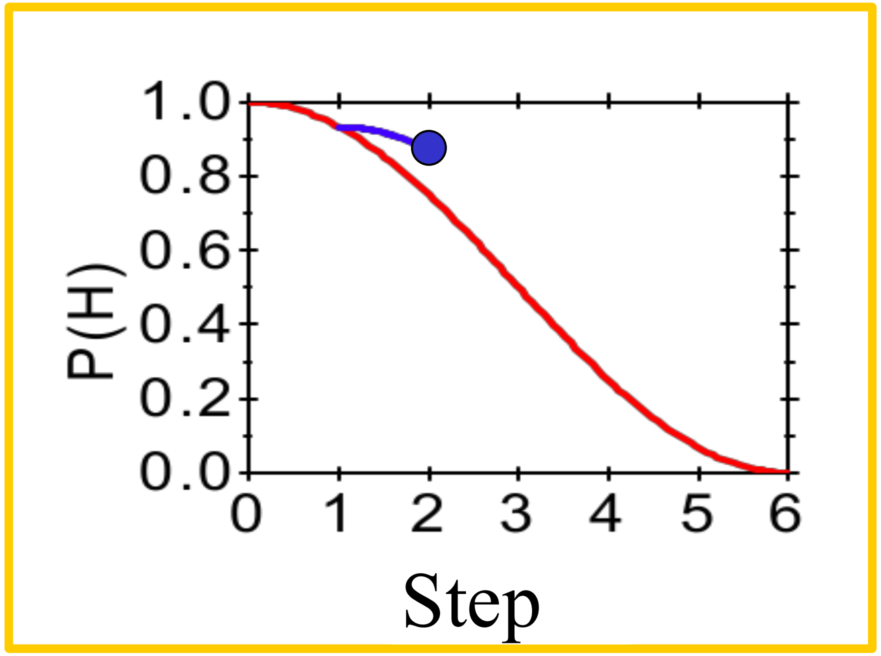

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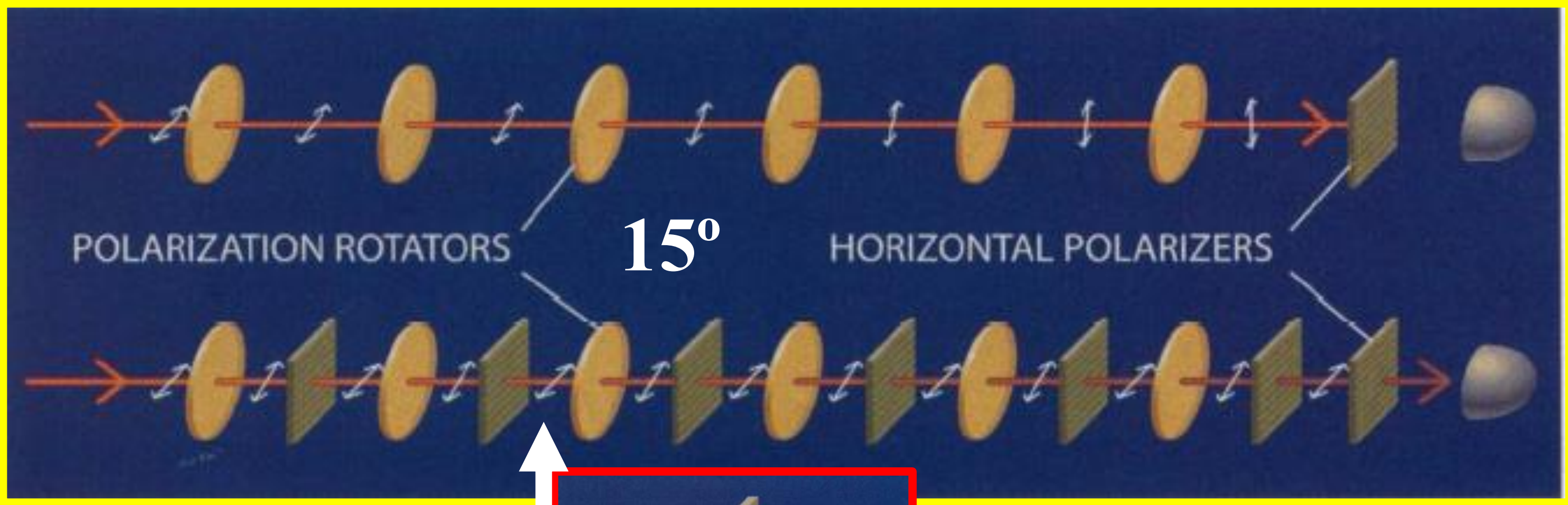

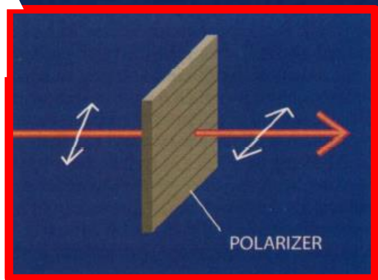

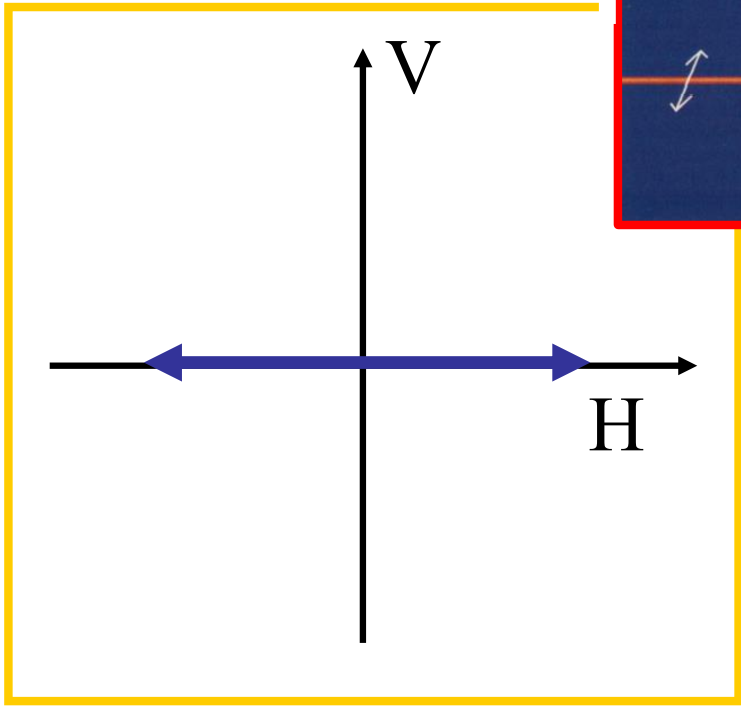

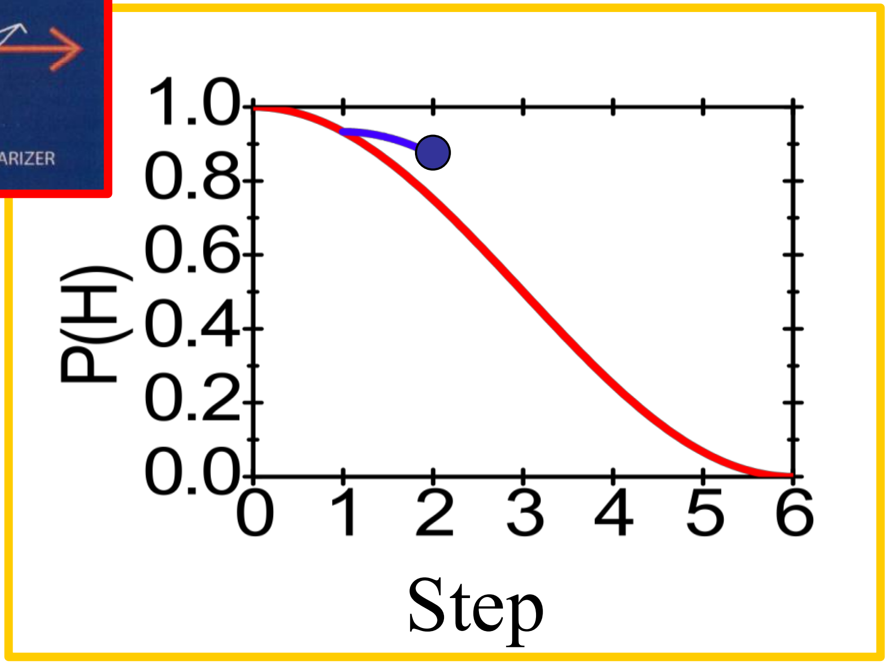

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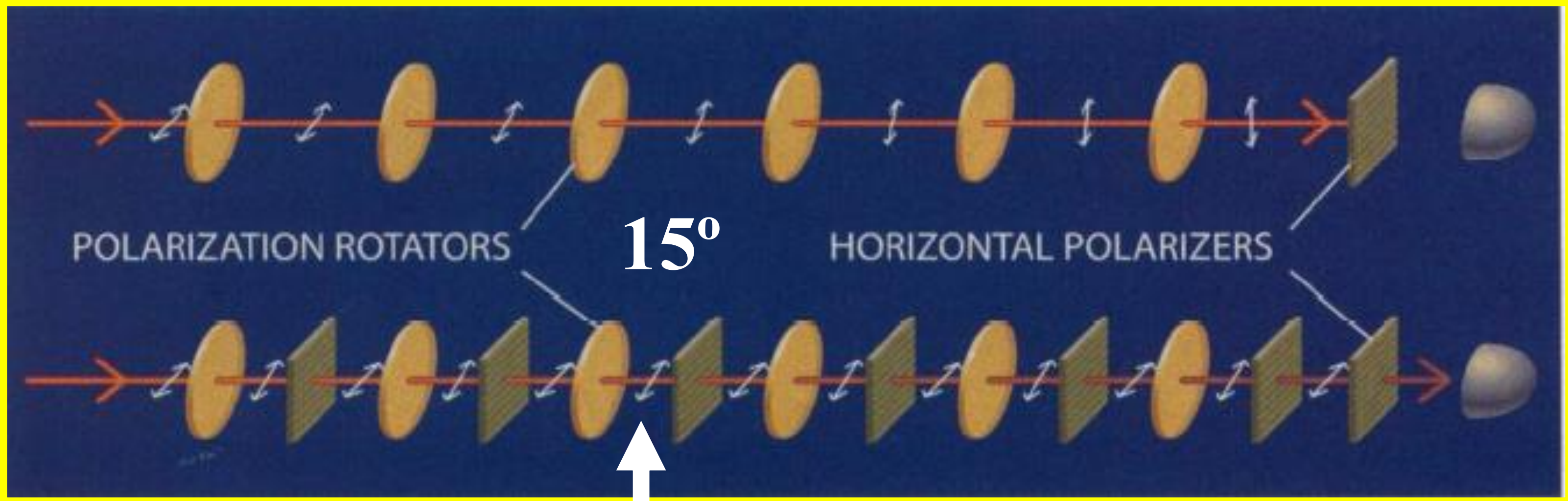

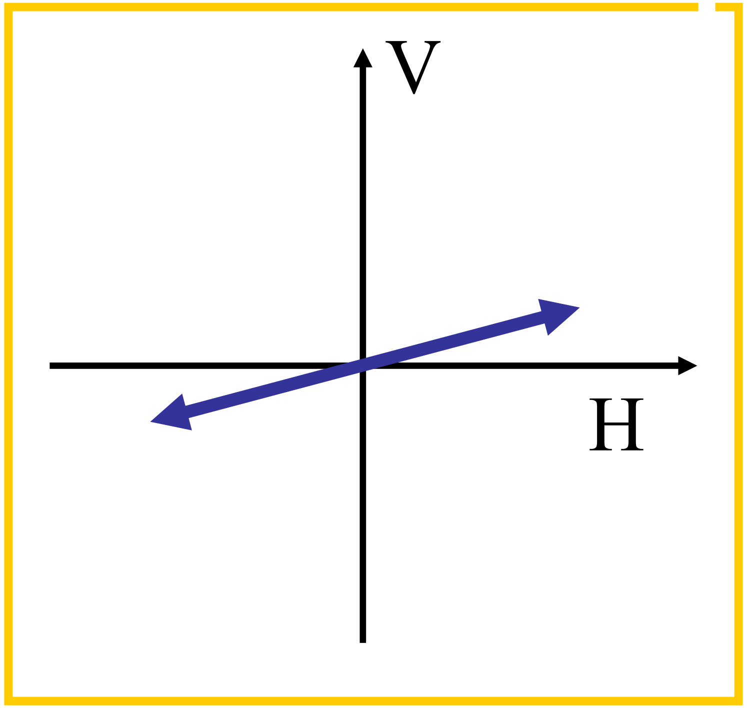

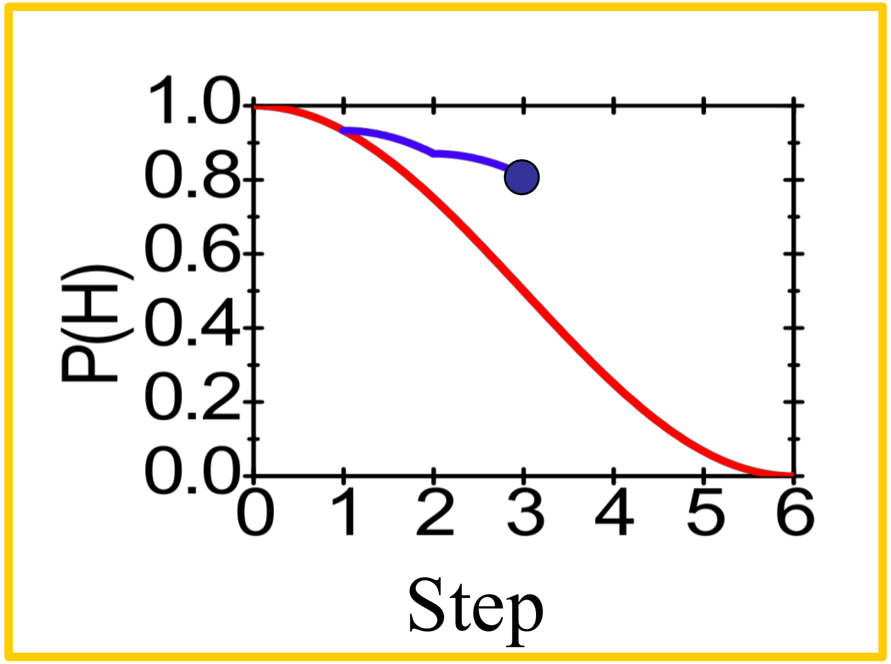

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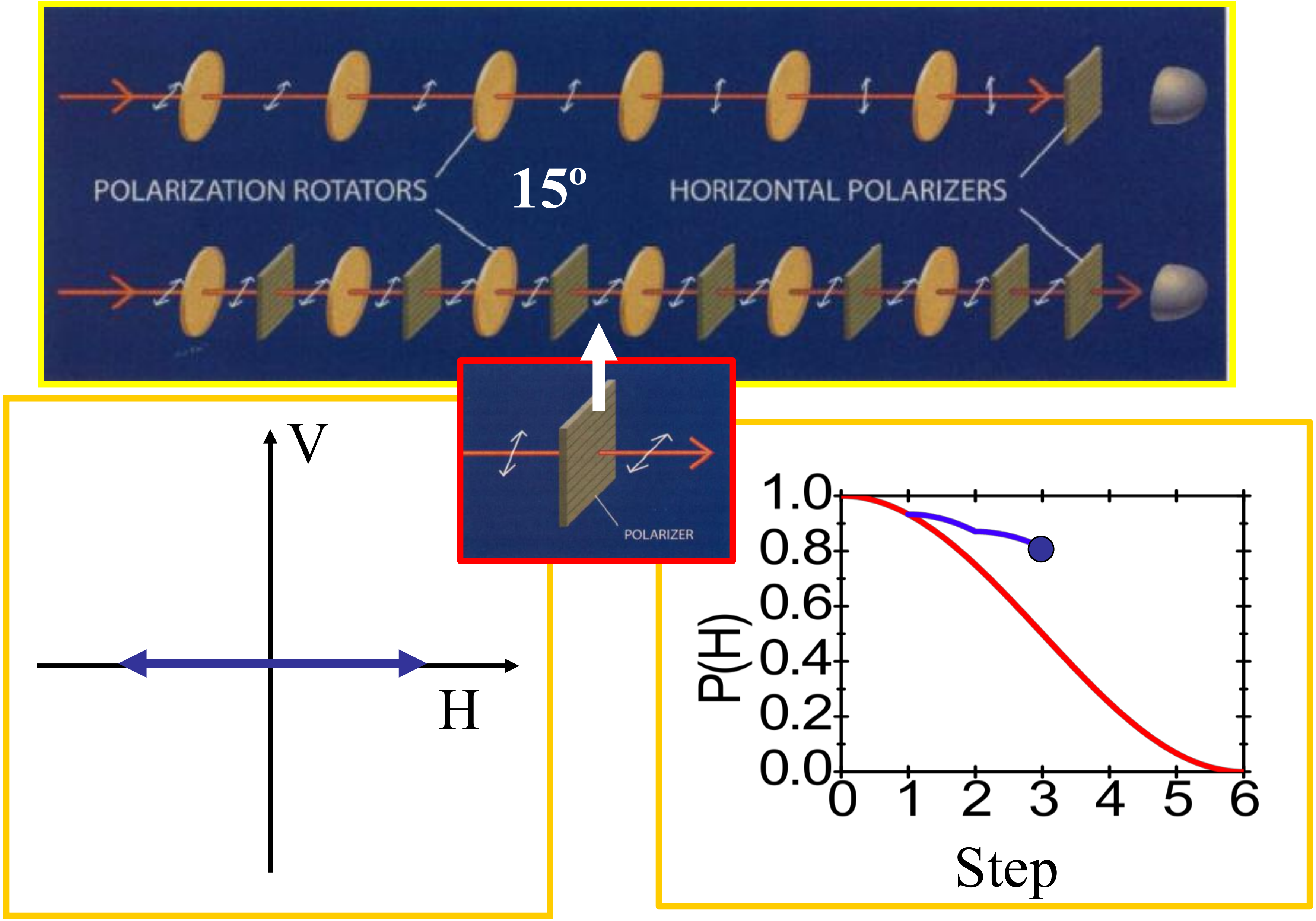

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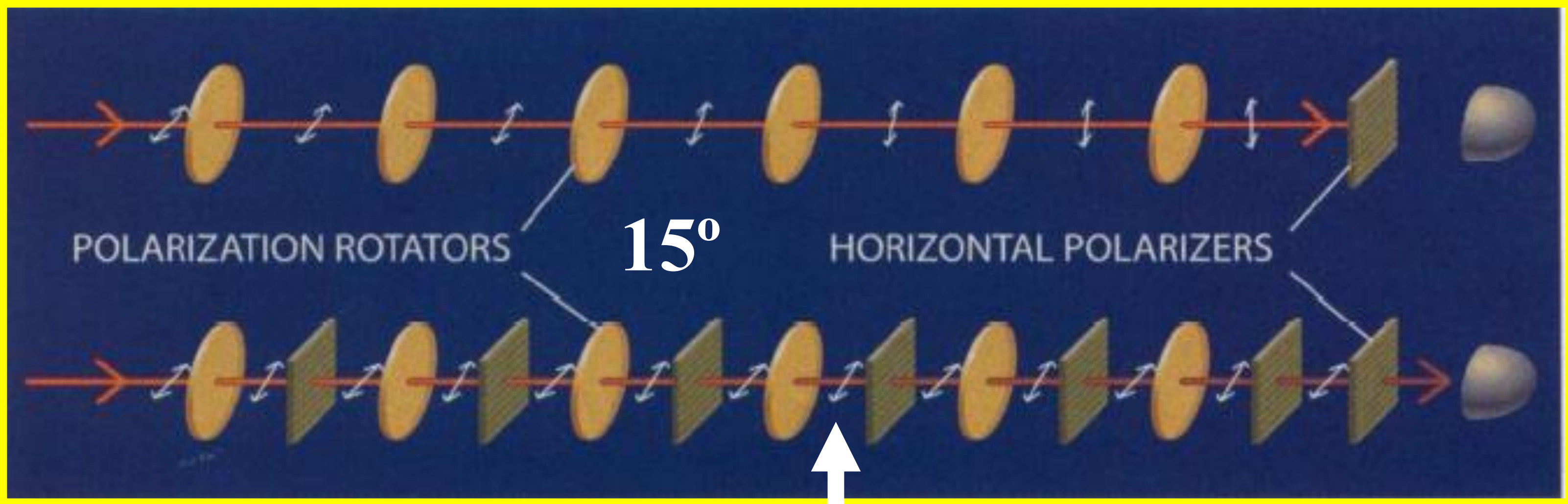

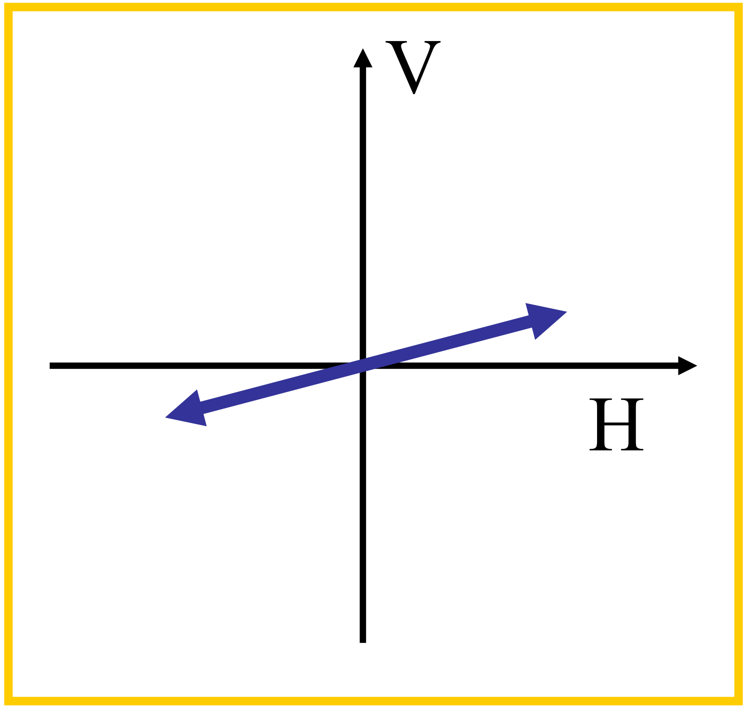

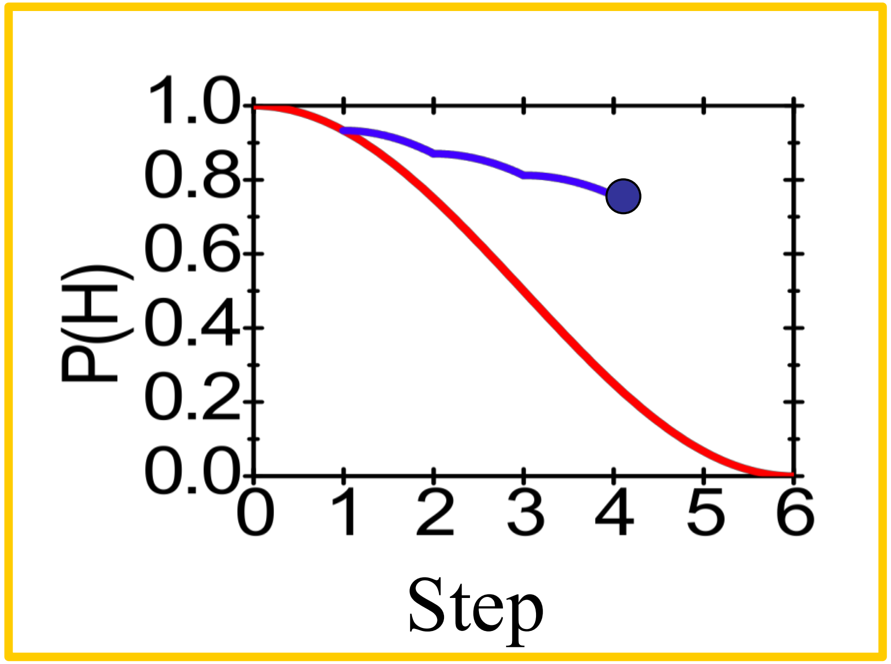

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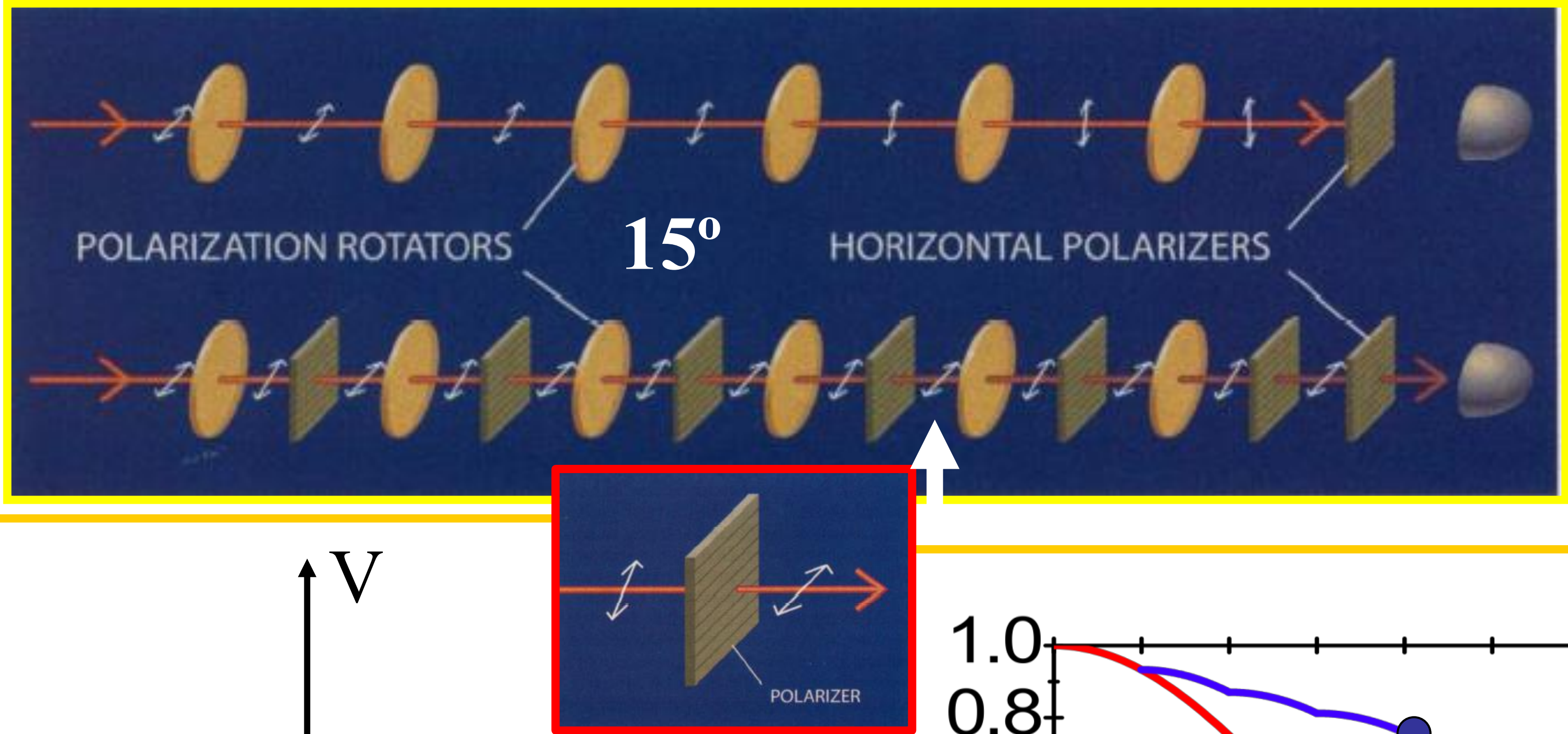

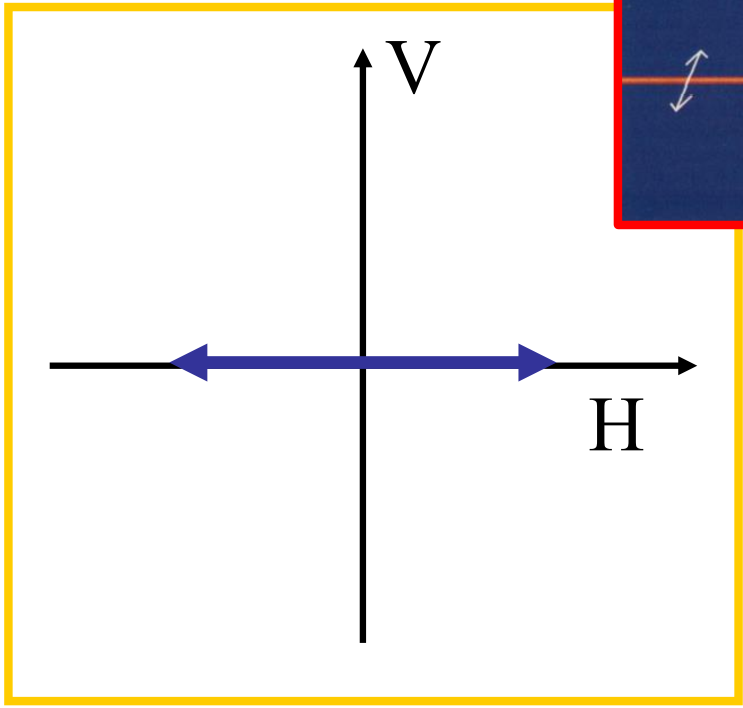

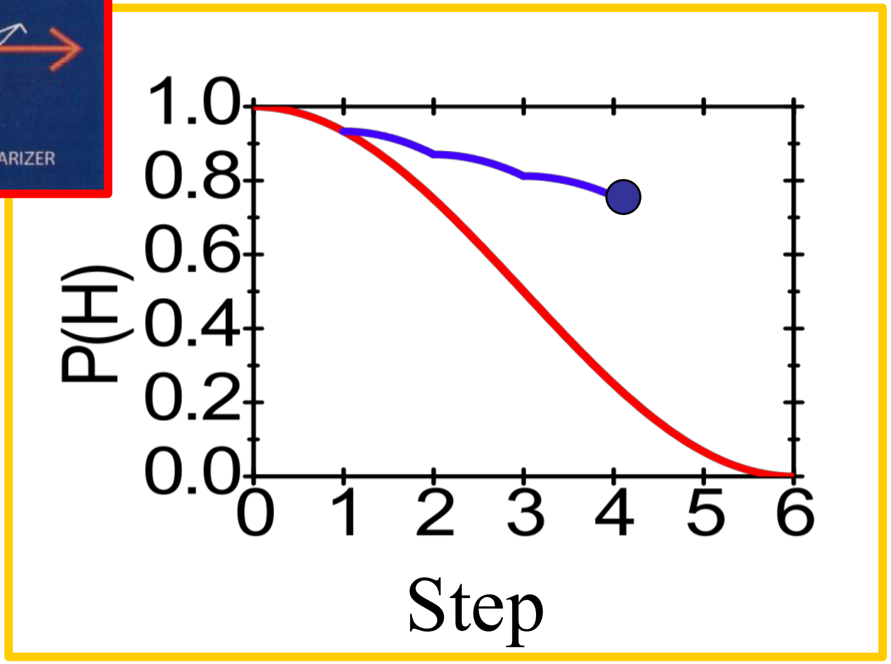

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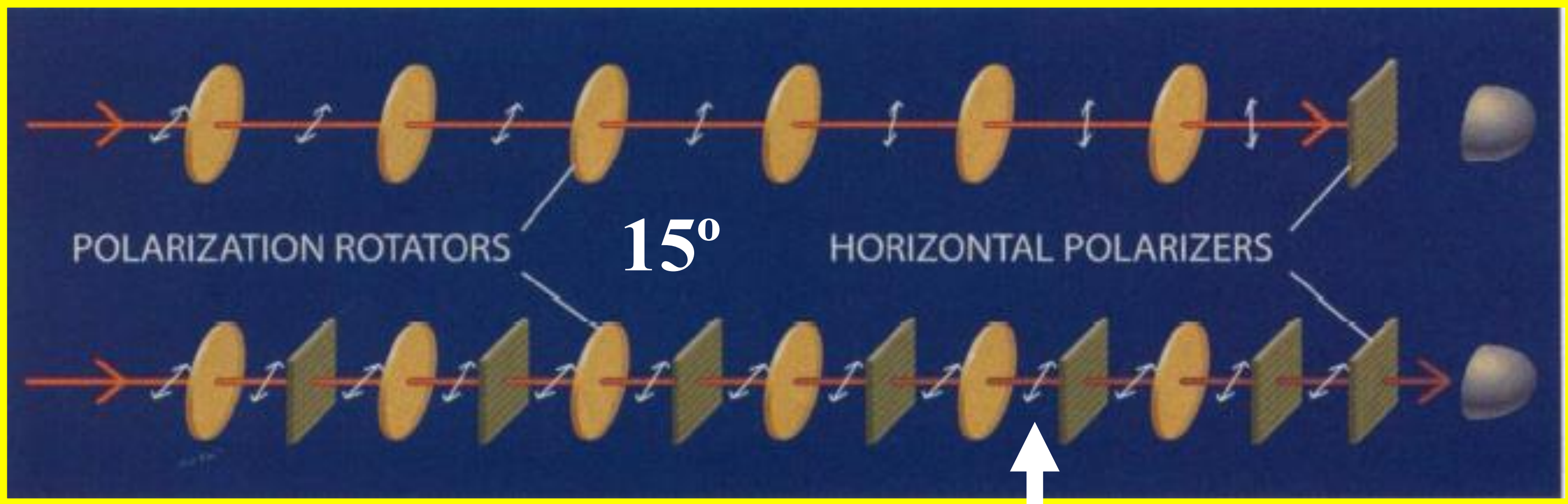

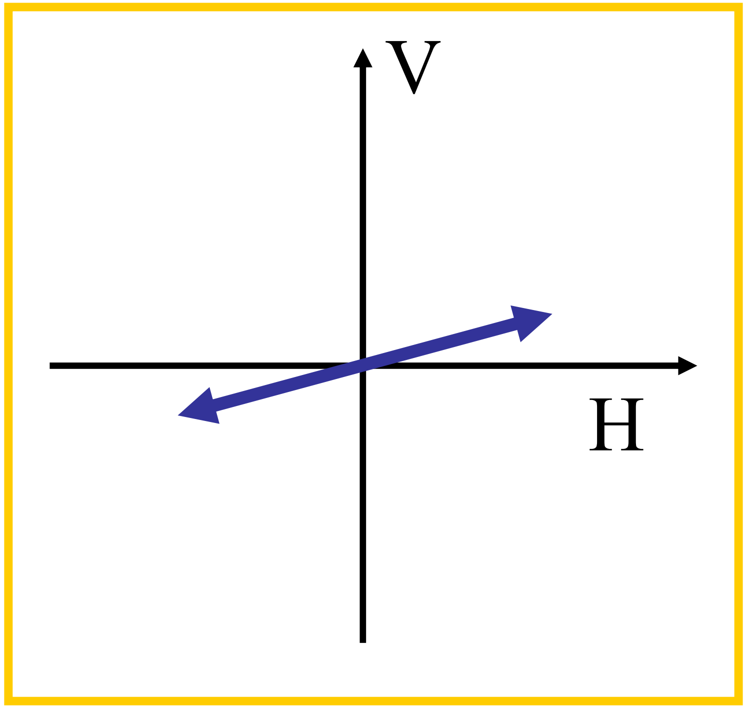

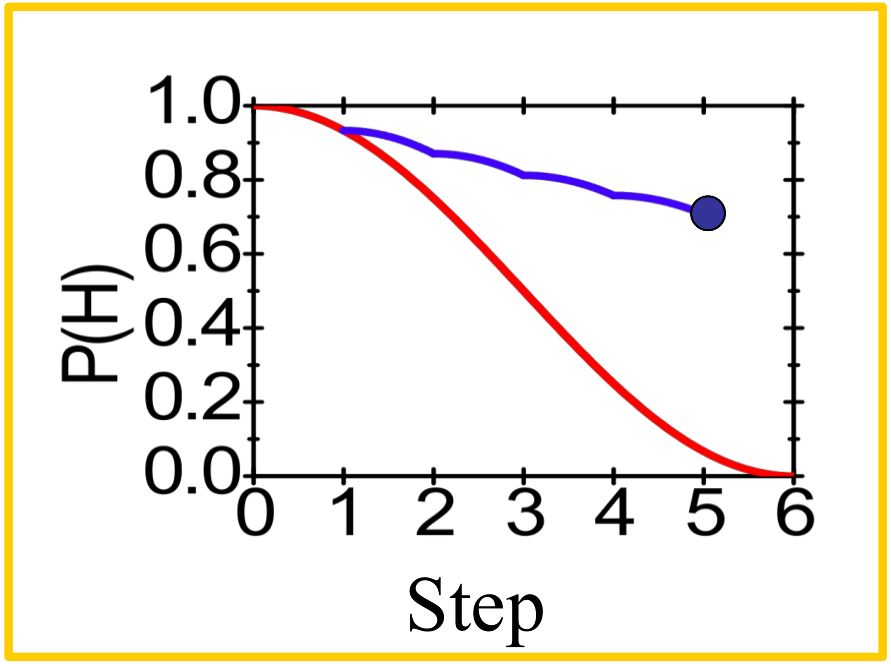

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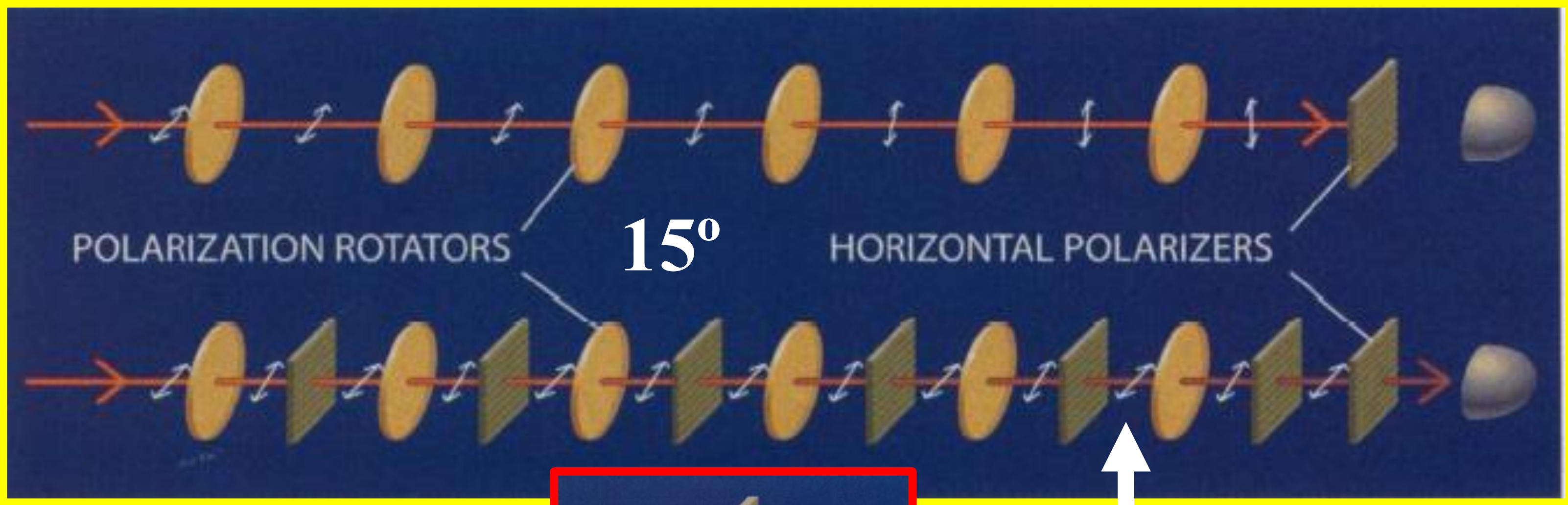

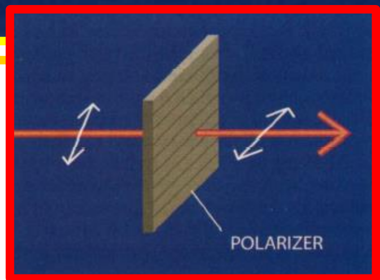

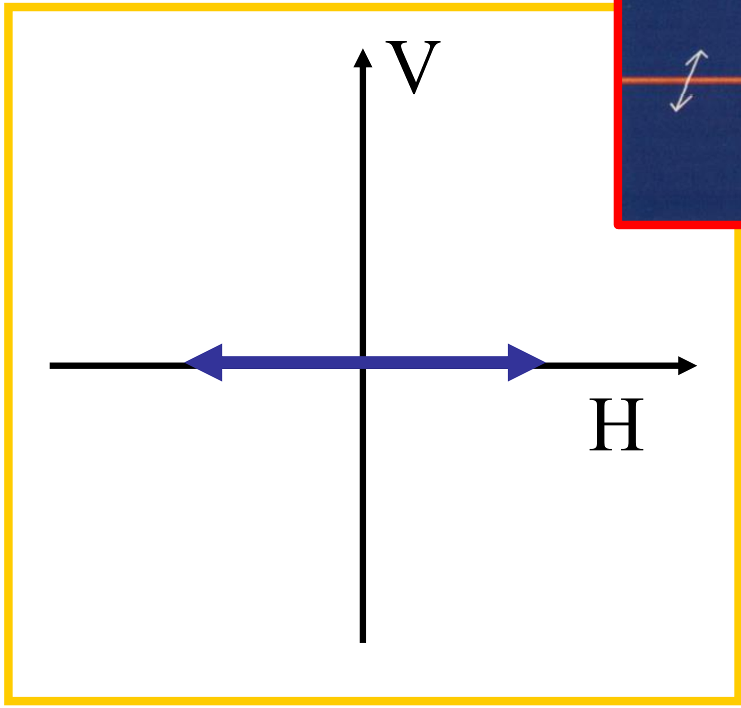

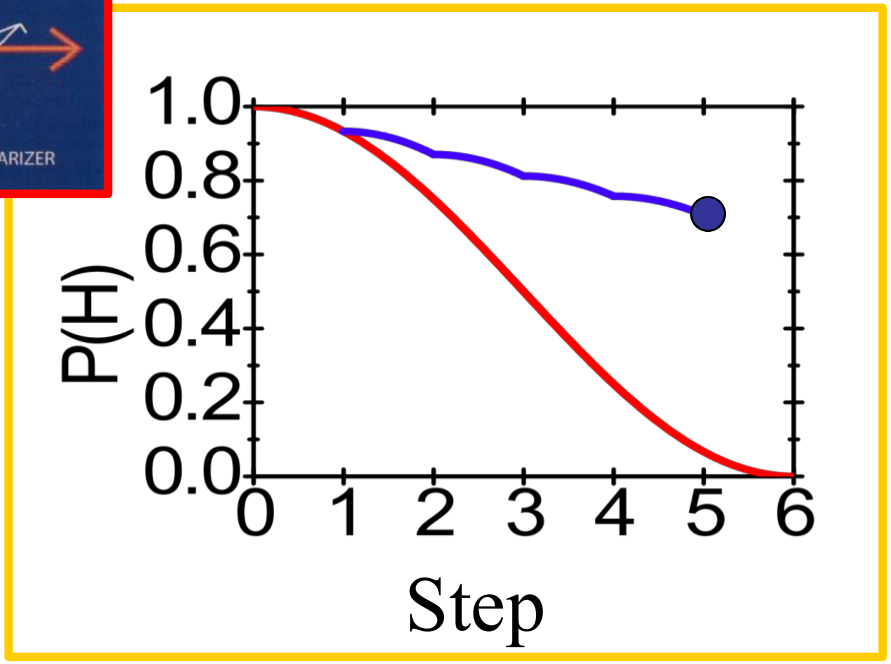

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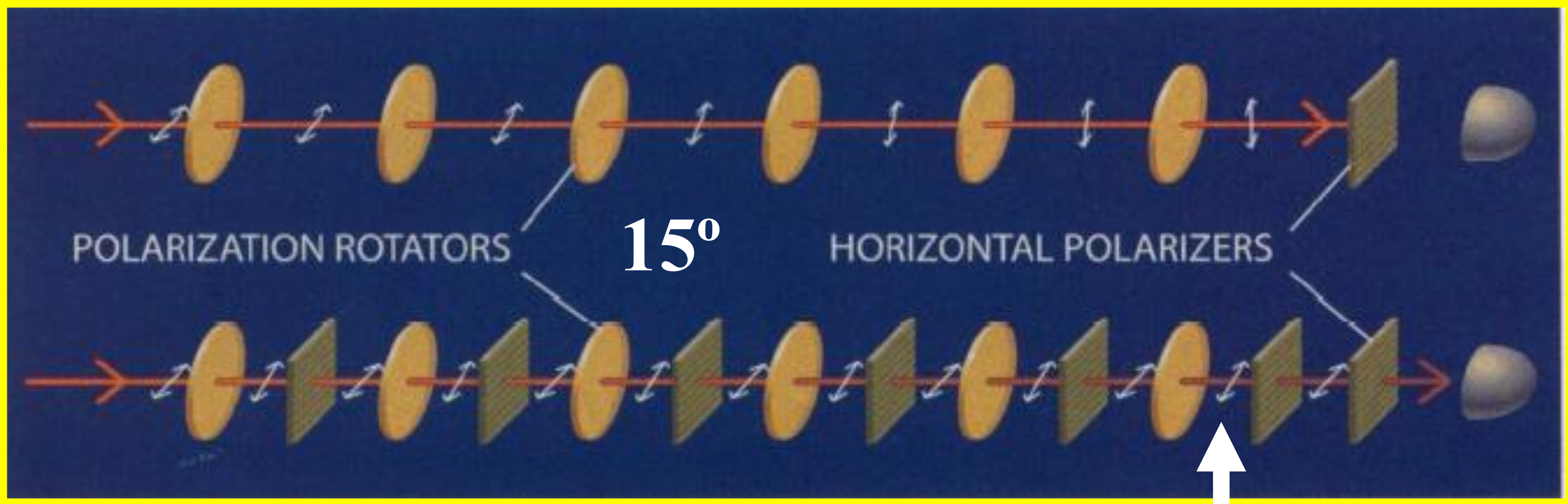

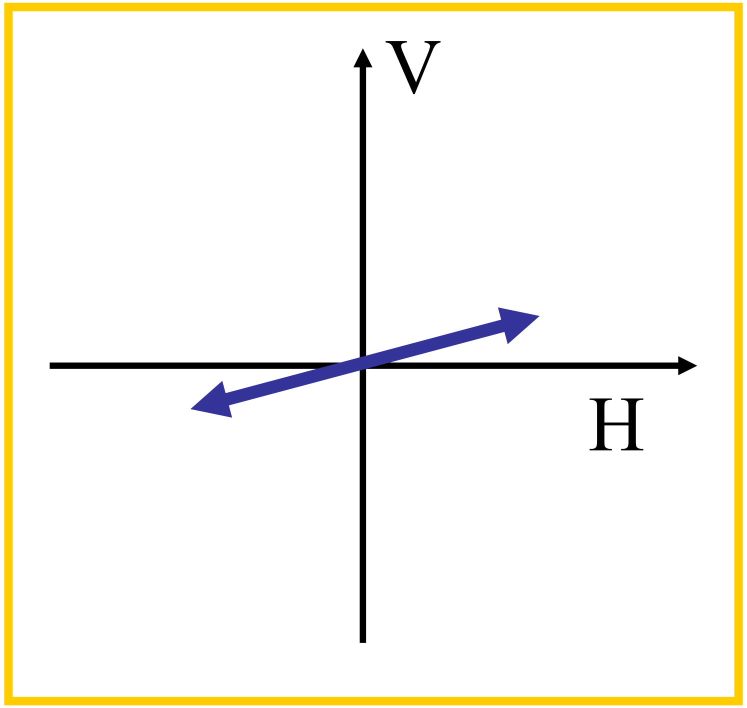

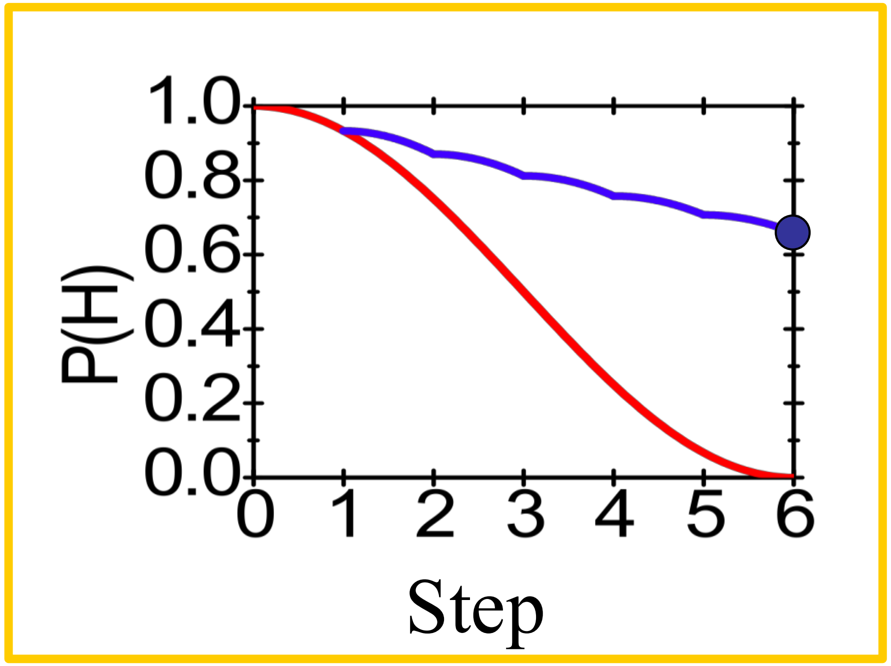

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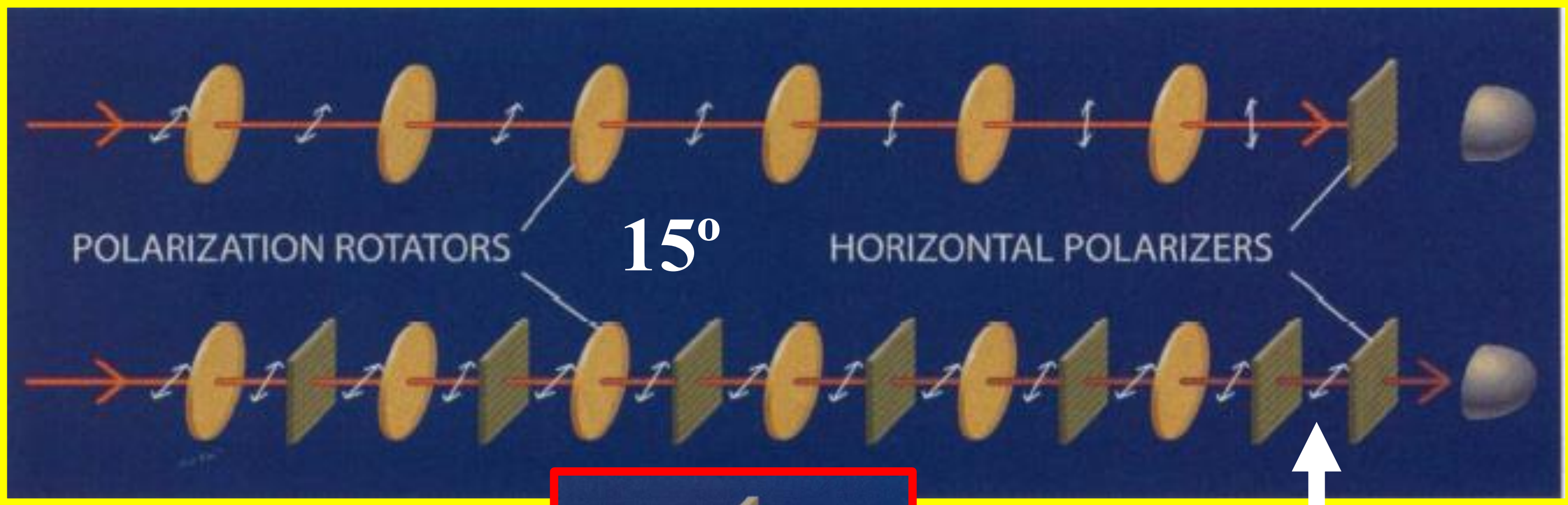

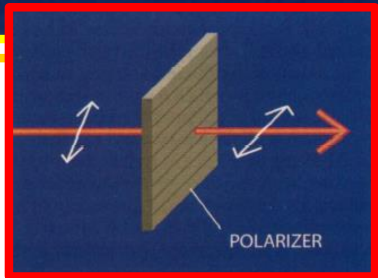

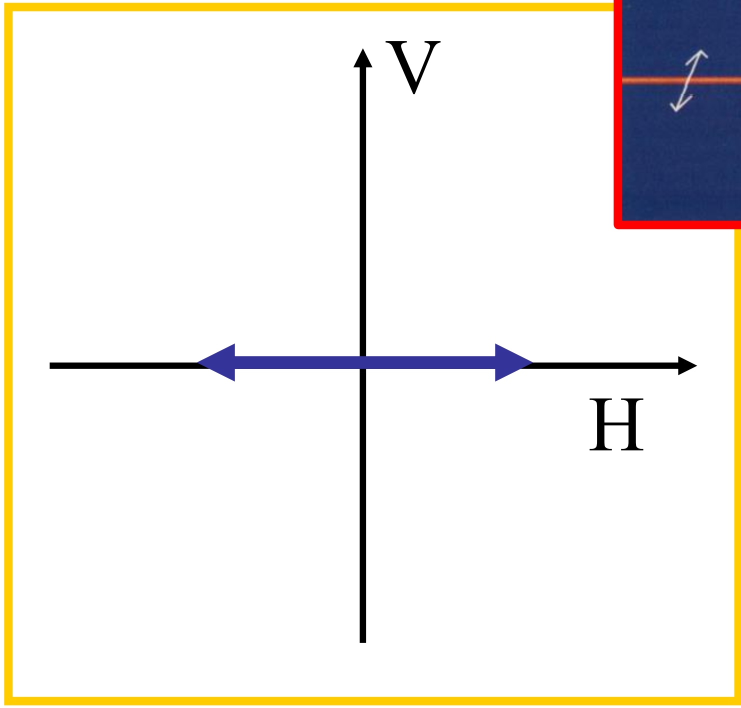

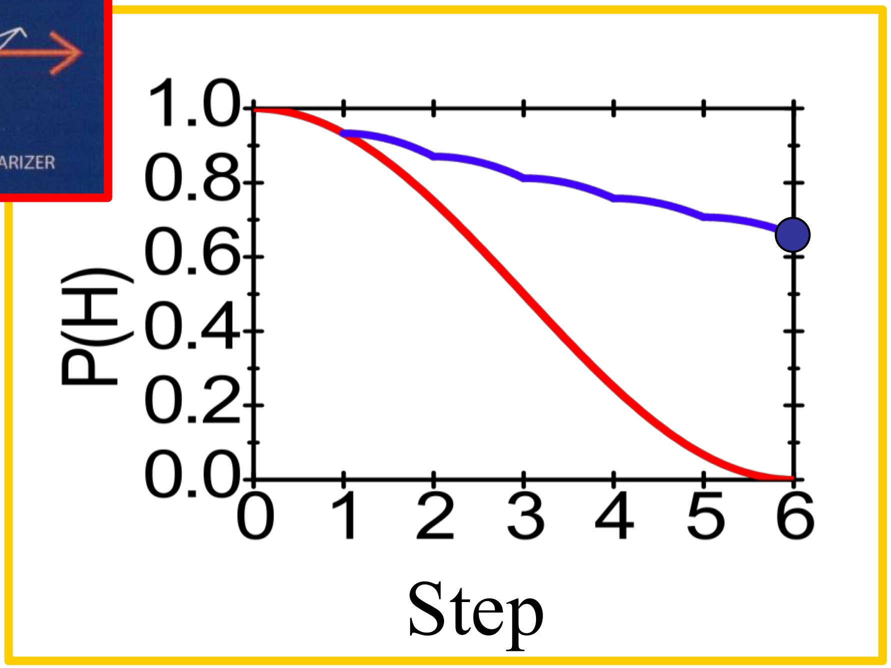

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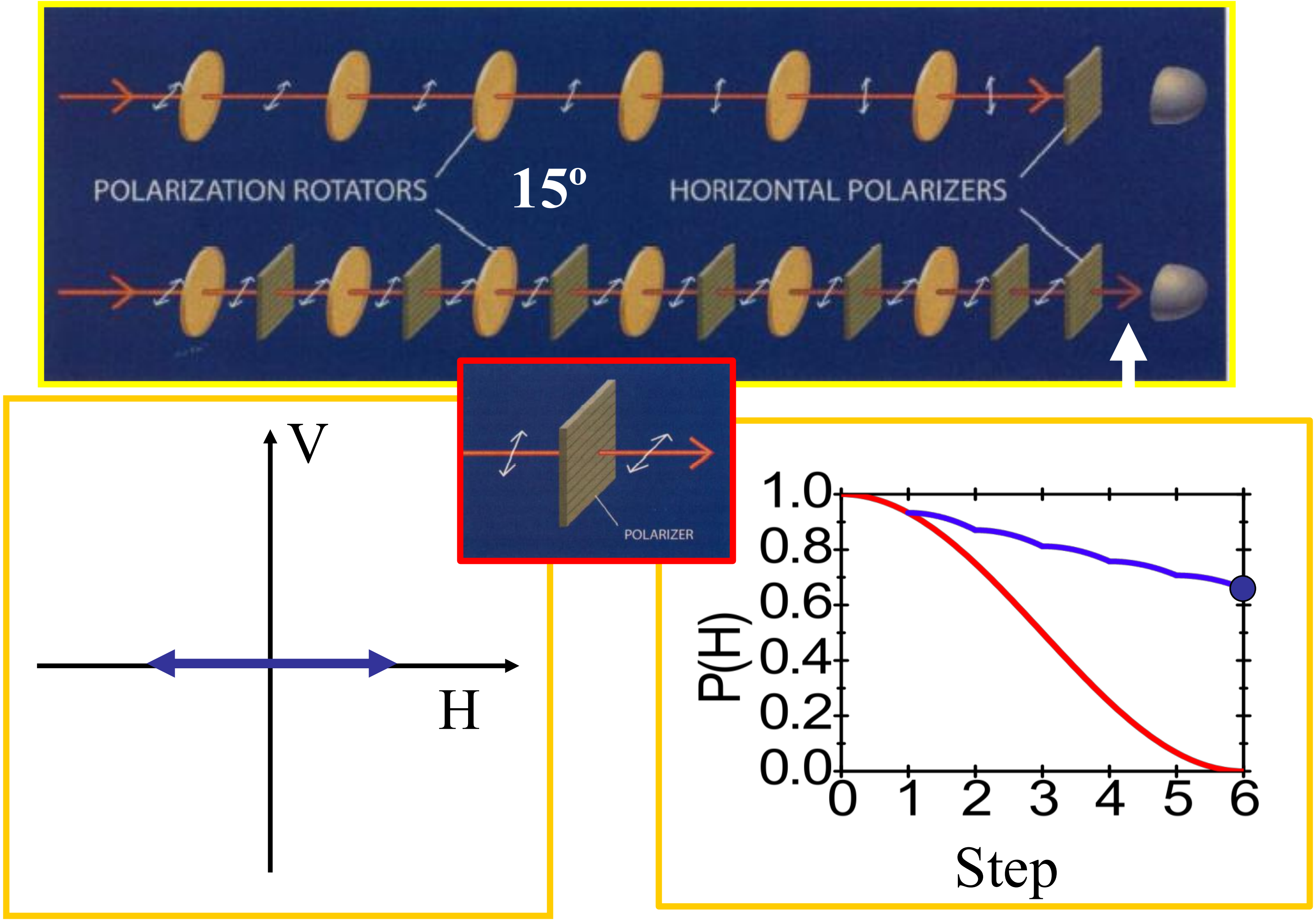

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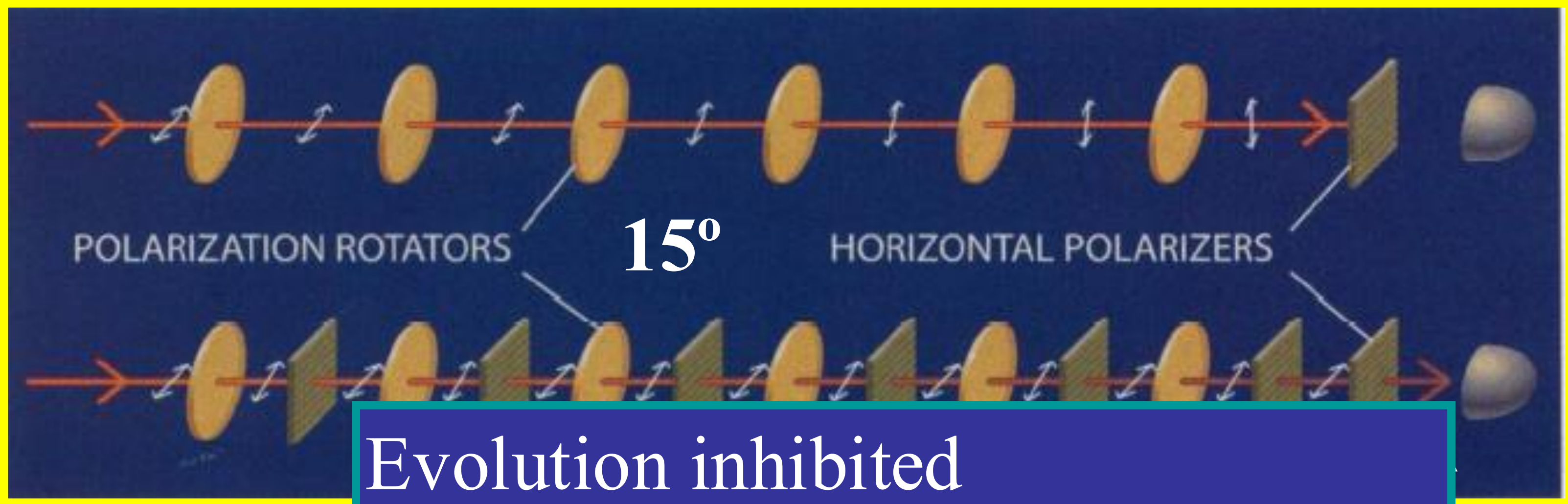
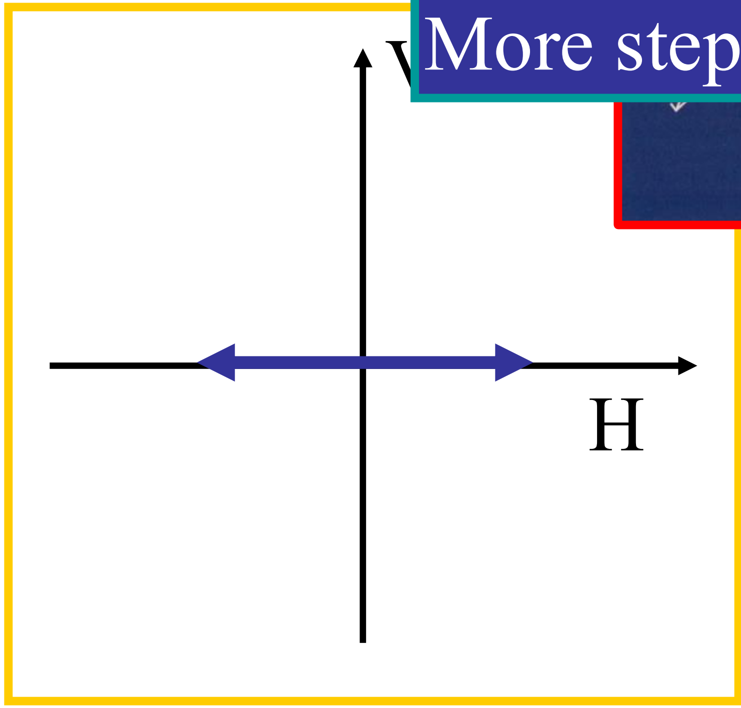
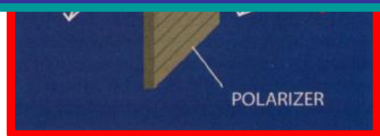
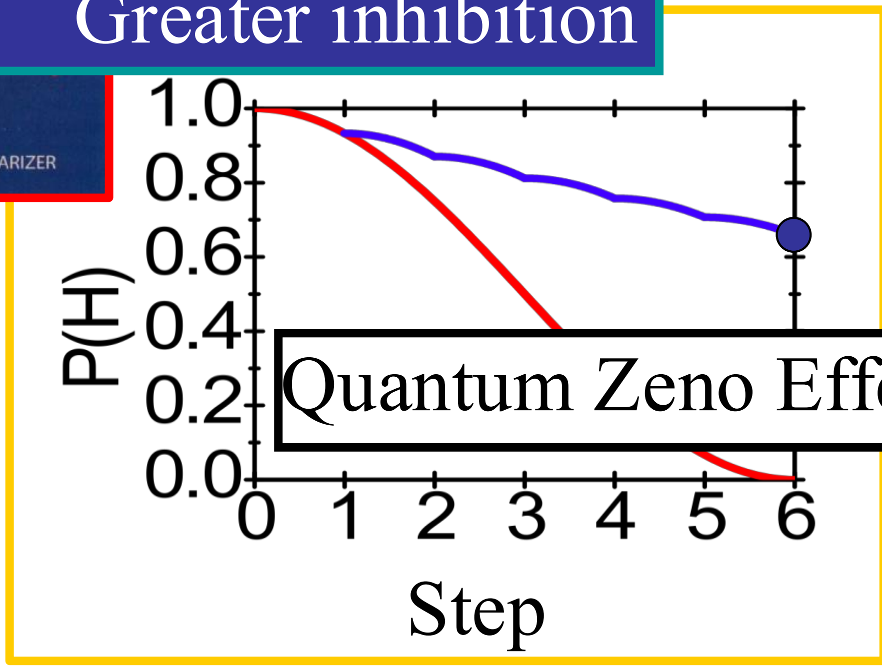


Evolution inhibited

More steps → Greater inhibition

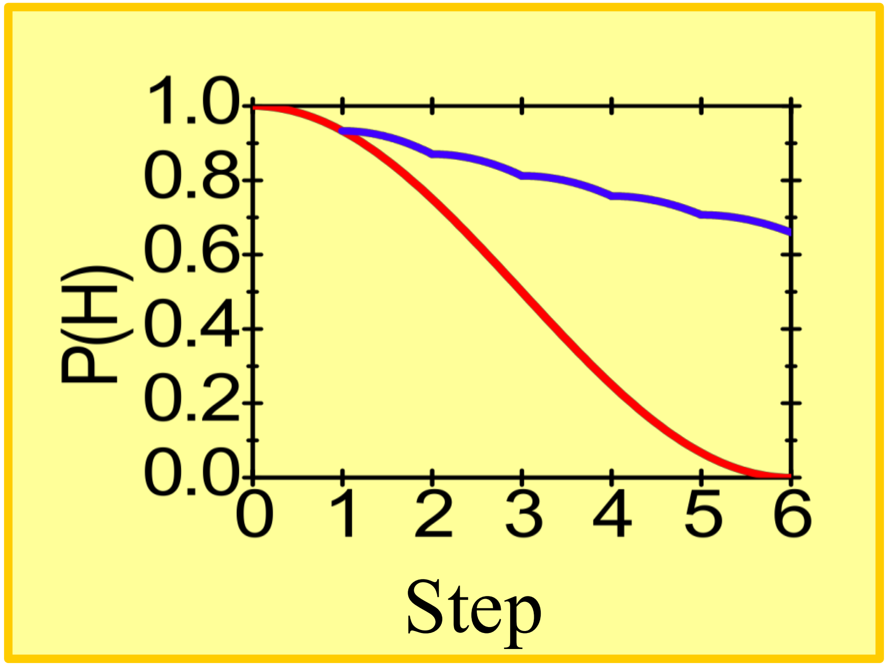






Quantum Zeno Effect

Stroboscopically measured TLS Dynamics



Quantum Zeno Effect

Exercises

2. **Decoherence: Pure Dephasing from a Stochastic Phase**

Consider a spin-$\frac{1}{2}$ particle initially polarized along the x-axis, i.e.,

$$\rho(0) = \frac{1}{2}I + S_x,$$

assuming $\hbar = 1$, and subject to a time-dependent stochastic phase $\delta(t)$ accumulated due to a fluctuating magnetic field along the z-axis. That is, $\delta(t)$ incorporates the effect of the field over time, and represents the net phase difference acquired between the spin states $|0\rangle$ and $|1\rangle$. In the interaction (rotating) frame, the time evolution of the density matrix is given by

$$\rho(t) = e^{-i\delta(t)S_z}\, \rho(0)\, e^{i\delta(t)S_z}.$$

We are interested in computing the normalized ensemble-averaged magnetization along the x-axis,

$$\langle S_x(t)\rangle = \frac{\mathrm{Tr}\left[\rho(t)S_x\right]}{\mathrm{Tr}\left[\rho(0)S_x\right]}.$$

(a) Show that

$$\langle S_x(t)\rangle = \langle \cos(\delta(t))\rangle,$$

assuming that $\delta(t)$ is a real-valued stochastic variable with zero mean.

Process Fidelity and Gate Comparison

1) Comparing Two Quantum Gates Consider

- U_1: **CNOT gate** with control qubit 1 and target qubit 2.
- U_2: a **flipped CNOT** gate with control qubit 2 and target qubit 1.

Both gates are represented by the following unitary matrices in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$$U_1 = \text{CNOT}_{1\to 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad U_2 = \text{CNOT}_{2\to 1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(a) Single-state analysis: Apply the gates U_1 and U_2 on the state $|00\rangle$. What can be concluded from this about the gates? Are they performing the same operation? Justify the answer, provide a counter-example if needed.

(b) Global comparison: Demonstrate if U_1 and U_2 are or are not equivalent quantum processes by computing the **process fidelity**

$$\mathcal{F} = \frac{\left|\text{Tr}(U_1^\dagger U_2)\right|}{\sqrt{\text{Tr}(U_1^\dagger U_1)}\sqrt{\text{Tr}(U_2^\dagger U_2)}}.$$

Interpret the result and explain what it tells you about the two gates.

Note: For unitary gates $\text{Tr}(U^\dagger U) = d$, where d is the Hilbert space dimension.

Quantum Error Correction: Bit Flip Code

2) Detecting and Correcting a Bit Flip Error Alice wants to transmit a qubit in the state $|\psi\rangle = a|0\rangle + b|1\rangle$ to Bob using a noisy quantum channel that occasionally flips individual qubits with probability p. To protect against a single bit-flip error, she encodes the logical qubit into three physical qubits as follows:

$$|\psi_0\rangle = (a|0\rangle + b|1\rangle) \otimes |00\rangle = a|000\rangle + b|100\rangle\,,$$

$$|\psi_1\rangle = \mathrm{CNOT}_{1\to 2}\,\mathrm{CNOT}_{1\to 3}\,|\psi_0\rangle = a|000\rangle + b|111\rangle\,.$$

Here $\mathrm{CNOT}_{i\to j}$ denote the CNOT gate with control qubit i and target qubit j. After transmission, Bob receives the state:

$$|\tilde{\psi}\rangle = a|001\rangle + b|110\rangle\,.$$

(a) Encoding. Explain why the state $|\psi_1\rangle = a|000\rangle + b|111\rangle$ is not a cloned state.

(b) Error Detection. Show that the operators $Z_1 Z_2$ and $Z_1 Z_3$ can be used to detect which qubit (if any) has flipped. Calculate the expectation values $\langle\tilde{\psi}|\, Z_1 Z_2\, |\tilde{\psi}\rangle$ and $\langle\tilde{\psi}|\, Z_1 Z_3\, |\tilde{\psi}\rangle$.

(c)Error Correction. Based on your results in (b), determine which qubit has flipped and apply the appropriate Pauli X operation to correct the error. Show that you recover the encoded state $|\psi_1\rangle$.

(d) Decoding Describe how Bob can decode the state to recover Alice's original qubit. Show the operations needed and verify that the final state is $(a|0\rangle + b|1\rangle) \otimes |00\rangle$.