Size Scaling of Decoherence Rates in Nearly-Decoherence Free Subspaces

Christopher S. Maierle* and Dieter Suter†
Department of Physics, University of Dortmund, D-44221 Germany
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In this work, we consider how decoherence rates scale with system size when qubits are subjected to spatially correlated random fields. We examine the size scaling of average decoherence rates for a number of encodings of logical qubits, all of which are decoherence free in the limit where each physical qubit sees the same fluctuating field. We show that imperfectly correlated fields give rise to the expected linear scaling of decoherence rates with system size but that the scaling rate varies depending on the encoding and the correlations. In particular, we show how the degree of spatial random field correlations seen by the qubits must be balanced against encoding efficiency (the ration of the number of logical to physical qubits) in order to optimize the size scaling of decoherence rates.

I. INTRODUCTION

As quantum computers grow to include more and more qubits, it becomes important to understand how decoherence rates scale with this increase of system size. In general, it is believed that decoherence rates increase linearly with number of qubits [1, 2]. Success probabilities of the form $e^{-\gamma t}$ where γ is a decoherence rate therefore scale exponentially with size. If, however, the interactions which give rise to decoherence display permutation symmetry so that groups of two or more qubits are coupled identically to the bath modes, it is then possible to encode qubits in decoherence free subspaces in which the qubits are immune to the effects of decoherence [3–6]. Any realistic system will of course contain terms which break the permutation symmetry. Nevertheless, if the symmetry breaking terms are small, the formerly decoherence free subspaces will still be sub-decoherent in the sense that states in these subspaces will tend to decohere more slowly than states chosen at random from the full Hilbert space [2, 4]. A large scale quantum computer is then constructed by concatenating together many such 'nearly- decoherence free' subspaces whereby one has considerable freedom in deciding how the component subspaces should be defined. In this work, we consider how the structure of the individual nearly-decoherence free subspaces affects the overall size scaling of decoherence rates.

Although we hope that the concepts presented here are generally applicable, we will mainly consider relaxation in the Markoffian regime where the resulting exponential relaxation allows for a simple and unambiguous characterization of relaxation rates. For the sake of simplicity, we will consider relaxation which can be modeled by fluctuating random fields. Previous work, however, has mainly focused on decoherence arising from qubits linearly coupled to a bosonic bath [1, 2, 5, 6] and so by way of connection we show in the first section how a bosonic bath can function as a random field and then briefly review the special cases of independent and collective decoherence. We conclude our introductory material by arguing that one can use an average decoherence rate as a measure of decoherence.

The main results of this work follow from analyzing the size scaling of average decoherence rates for a number of encodings, all of which are decoherence free in the limit where each physical qubit sees the same fluctuating field. We show that imperfectly correlated fields give rise to the expected linear scaling of decoherence rates with system size but that the scaling rate varies depending on the encoding and the correlations. In particular, we show how the degree of spatial random field correlations seen by the qubits must be balanced against encoding efficiency (the ration of the number of logical to physical qubits) in order to optimize the size scaling of decoherence rates.

II. DECOHERENCE MODELS

We will consider relaxation which can be understood as arising from a fluctuating random field. In this model, the Hamiltonian is,

$$H = \sum_i \frac{\omega_i}{2}\sigma_z^{(i)} + \sum_i \gamma_i \mathbf{B}^{(i)}(t)\cdot\sigma^{(i)} \qquad (1)$$

where $\mathbf{B}^{(i)}(t)$ is the fluctuating field seen by the ith qubit and the σ are Pauli spin operators. In order that the above Hamiltonian give rise to relaxation we must consider an ensemble of quantum computers, each of which is subject to a different set of fields. We then define an average density matrix, $\rho_{avg} = \overline{U(t,0)\rho(0)U^\dagger(t,0)}$ where $U(t,0)$ is the time evolution operator generated by the Hamiltonian in Eq. (1) and the bar indicates an average over the ensemble of random fields. Physically, the ensemble averaging arrises either because we perform many repeated experiments and average the results, or because each measurement corresponds to an average over many individual quantum computers, as is the case in ensemble NMR quantum computing. Regardless of the underlying physical nature, there exists, in general, no closed form solution for ρ_{avg}. However, in the Markoffian limit, where the fields fluctuate on a timescale fast compared

*Electronic address: chris@e3.physik.uni-dortmund.de
†Electronic address: Dieter.Suter@physik.uni-dortmund.de

to the dynamics which they generate, one can write the approximate equation of motion [7],

$$\dot{\rho}_{avg} = -i[\sum_i \frac{\omega_i}{2}\sigma_z^{(i)}, \rho_{avg}] - \Gamma\rho_{avg} \quad (2)$$

where the relaxation superoperator, Γ, is given by

$$\Gamma\rho = \sum_{i,j}\sum_{q,q'} \frac{1}{2} J_{qq'}^{ij}(q\omega_i)[\sigma_{q'}^{(j)}, [\sigma_q^{(i)}, \rho]] \quad (3)$$

and the spectral densities $J_{qq'}^{ij}(\omega)$ are

$$J_{qq'}^{ij}(\omega) = \gamma_i\gamma_j \int_{-\infty}^{\infty} \overline{B_q^{(i)}(t)B_{q'}^{(j)}(t+\tau)} e^{i\omega\tau} d\tau \quad (4)$$

where the indices i and j label qubits and q and q' take on the values 0 and ± 1 corresponding to σ_z and $\frac{1}{\sqrt{2}}\sigma_\pm$ respectively.

If the random fields do not fluctuate fast enough to induce transitions between the Zeeman levels of $\sum_i \frac{\omega_i}{2}\sigma_z^{(i)}$, then the dominant source of relaxation is the $[\sigma_z^{(i)}, [\sigma_z^{(j)}, \rho_{avg}]]$ term in Eq. (3). In this case, often known as pure dephasing, relaxation can be understood as arising from fluctuations in energy levels caused by the z component of the random field.

As mentioned in the introduction, previous investigations of decoherence in quantum computers have often utilized the spin boson model as a starting point, rather than the semi-classical fluctuating random field model outlined above. When using the spin boson model, one starts with the Hamiltonian, $H_{SB} = H_S + H_B + H_I$ where $H_S = \sum_i \frac{\omega_i}{2}\sigma_z^{(i)}$ is the Hamiltonian for the qubits, $H_B = \sum_k \omega_k^B (b_k^\dagger b_k + \frac{1}{2})$ is the Hamiltonian for the bath modes, and the interaction between the bath and the qubits is given by,

$$H_I = \sum_k g_k\sigma_+ b_k + f_k\sigma_+ b_k^\dagger + h_k\sigma_z b_k + \text{h.c.} \quad (5)$$

One then calculates a reduced density matrix $\rho_{red} = \text{Tr}_B[\rho]$ by tracing over the bath degrees of freedom in the full density matrix. As with the semi-classical random field model, there exists, in general, no closed form solution for the reduced density matrix. One can, again, consider the Markoffian limit where the bath fluctuates on a timescale fast compared to the dynamics that it generates. If one assumes, as is usually done, an initial state for the system and bath of the form $\rho_S(0) \otimes \rho_B(0)$ so that the system and bath are not initially entangled, then Eq. (2) also holds for the reduced density matrix, ρ_{red} when the classical correlation functions $\overline{B_q^{(i)}(t)B_{q'}^{(j)}(t+\tau)}$ are replaced by their quantum mechanical analogs $\text{Tr}_B[\hat{B}_q(t)\hat{B}_{q'}(t+\tau)\rho_B(0)]$ where the operators $\hat{B}_q(t)$ are constructed from the bath operators in Eq. (5) written in the interaction representation.

There are, of course, important differences between the fully quantum mechanical spin boson model and the semi-classical random field model. Unless ad-hoc modifications are made, semi-classical relaxation models generally lead to infinite temperature equilibrium states where all energy levels are equally populated whereas fully quantum mechanical models can yield the correct thermodynamic equilibrium states [7]. Additionally, fully quantum mechanical models account for entanglement between the system and bath and can therefore lead to decoherence even at zero temperature where the ensemble of bath states contains only one member [2]. Nevertheless, assuming that the system and bath are initially unentangled, the equations of motion for ρ_{avg} and ρ_{red} depend identically on correlation functions of the fields and field operators. We will not prove this statement here, but it holds generally regardless of whether the dynamics are Markoffian or not.

III. COLLECTIVE AND INDEPENDENT DECOHERENCE

We now briefly review the special cases of independent and collective decoherence. In the Markoffian limit, relaxation rates have a very simple dependence on the spectral densities $J_{qq'}^{ij}(\omega)$, which are Fourier transforms of correlation functions for the random fields seen by the ith and jth qubits. If there exists no correlation between the random fields seen by different qubits, then the spectral densities $J_{qq'}^{ij}(\omega)$ vanish unless $i = j$ and we can rewrite the relaxation superoperator Γ as a sum of superoperators each of which determines the relaxation of a single physical qubit. This case is known as independent decoherence because each qubit relaxes independently of the state of the others [1, 2].

If, on the other hand, each qubit is affected identically by the fluctuating field, then the Hamiltonian in Eq. (1) can be rewritten in the form,

$$H = \sum_i \omega_i\sigma_z^{(i)} + \gamma B(t)\cdot\sigma^{\text{coll}} \quad (6)$$

where σ^{coll} is a collective operator with components $\sigma_q^{\text{coll}} = \sum_i \sigma_q^{(i)}$. In this case, known as collective decoherence [5, 6], one can construct a decoherence free subspace out of the states which are annihilated by the collective operators σ_q^{coll}. These subspaces are decoherence free [4] because, within the subspace all matrix elements of the interaction term $\gamma B(t)\cdot\sigma^{\text{coll}}$ are zero so that the fluctuating field effectively does not exist so long as we stay within the subspace.

Since it is hard to imagine why a large number of qubits would all be subject to the exact same bath fluctuations, previous workers have proposed encoding logical qubits in small clusters of physical qubits [8, 9]. The idea is that it may be reasonable to expect a small group of qubits to see nearly the same random field fluctuations if the qubits are located physically close to one another. In other words, when it is not possible to write the Hamiltonian, even

approximately, in terms of collective operators involving all the physical qubits, one hopes for a Hamiltonian which can be written in the form,

$$H = \sum_i \omega_i \sigma_z^{(i)} + \sum_c \gamma B^c(t) \cdot \sigma^c \tag{7}$$

where the σ^c are collective operators for small clusters of spins and have the components $\sigma_q^c = \sum_{i \in c} \sigma_q^{(i)}$ with the sum running over physical qubits in the cluster labeled by c. Again, the states which are annihilated by the cluster operators σ_q^c can be used to form a decoherence free subspace. Our primary goal is to consider decoherence which arrises inside of these subspaces due to the presence of imperfectly correlated fields. We will find it useful to have a simple measure of decoherence and so to this end, we argue in the next section that the average decoherence rate represents a good measure of 'overall' decoherence.

IV. AVERAGE DECOHERENCE RATE AS A MEASURE OF DECOHERENCE

A system of n qubits inhabits a Liouville space of dimension 4^n and, rather than focusing on the decay of individual matrix elements, we would like to consider the overall loss of quantum information due to decoherence. In order to quantify this we can consider the overlap $F(t)$ between the actual state of the quantum computer at time t and the state that would result in the absence of decoherence. Taking the operations of the ideal quantum computer to be contained in the Liouville operator $\mathcal{L}(t)$ and the decoherence to be represented by the superoperator Γ, we write,

$$F(t) = \langle\langle \rho_0 | \mathcal{U}_{\mathcal{L}}^\dagger \mathcal{U}_{\mathcal{L}+\Gamma} | \rho_0 \rangle\rangle = \langle\langle \rho_0 | \mathcal{U}_{\Gamma(t)} | \rho_0 \rangle\rangle \tag{8}$$

where $\mathcal{U}_{\mathcal{L}}$, $\mathcal{U}_{\mathcal{L}+\Gamma}$, and $\mathcal{U}_{\Gamma(t)}$ are the Liouville space time evolution operators generated by $\mathcal{L}(t)$, $\mathcal{L}(t)+\Gamma$, and $\mathcal{U}_{\mathcal{L}} \Gamma \mathcal{U}_{\mathcal{L}}^{-1}$ respectively and ρ_0 is the initial state of the quantum computer. The double brackets in the above expression indicate a scalar product in Liouville space which corresponds to the quantity $\mathrm{Tr}[\rho_0 U_{\Gamma(t)} \rho_0 U_{\Gamma(t)}^\dagger]$ in the usual Hilbert space notation [10]. From a physical point of view, one expects that as the quantum computer functions, it will transfer states through various Liouville space pathways. If this transfer is ergodic in the sense that each pathway tends to equally sample the various possible relaxation rates, then we expect that $F(t)$ will decay with a rate which is an average of the decoherence rates of the system. We can present a somewhat firmer derivation of this concept by using the projection operator technique [11] to write the following equation of motion for $F(t)$:

$$\dot{F}(t) = -\langle\langle \rho_0 | \Gamma(t) | \rho_0 \rangle\rangle F(t) - \int_0^t K(t,\tau) F(\tau) d\tau \tag{9}$$

The kernel $K(t,\tau)$ is given by,

$$K(t,\tau) = \langle\langle \rho_0 | \Gamma(t) \mathcal{U}_{(1-P)\mathcal{U}_{\mathcal{L}} \Gamma \mathcal{U}_{\mathcal{L}}^{-1}} (1-P) \Gamma(\tau) | \rho_0 \rangle\rangle \tag{10}$$

where P is the projection operator $|\rho_0\rangle\rangle\langle\langle\rho_0|$ and we have assumed the normalization condition $\langle\langle \rho_0 | \rho_0 \rangle\rangle = 1$. If we assume that the computer functions on a timescale τ_{QC} much faster than the timescale for decoherence ($\tau_{QC}|\Gamma| \ll 1$), as it must if the quantum computer is able to successfully calculate, then we expect $K(t,\tau)$ to go to zero for times $t - \tau \gg \tau_{QC}$ so that the integral $\int_0^t K(t,\tau) F(\tau) d\tau$ is of the order $|\Gamma(t)|^2 \tau_{QC} F(t)$. The second term is therefore small compared to the first (which is of order $|\Gamma(t)| F(t)$) and so we drop it. Solving for $F(t)$ then yields,

$$F(t) = e^{-\int_0^t \langle\langle \rho_0 | \Gamma(\tau) | \rho_0 \rangle\rangle d\tau} F(0) \tag{11}$$

The eigenvalues of Γ give the decoherence rates of the system and since $\Gamma(t) = \mathcal{U}_{\mathcal{L}} \Gamma \mathcal{U}_{\mathcal{L}}^{-1}$ is a unitary transformation of Γ, it must have the same eigenvalues. Defining $|\psi_n(t)\rangle\rangle$ as the normalized Liouville space eigenvectors of $\Gamma(t)$ with eigenvalues γ_n, we can write,

$$\int_0^t \langle\langle \rho_0 | \Gamma(\tau) | \rho_0 \rangle\rangle d\tau = \int_0^t \sum_n |c_n(\tau)|^2 \gamma_n d\tau \tag{12}$$

where the $c_n(\tau)$ are given by $\langle\langle \rho_0 | \psi_n(\tau) \rangle\rangle$. The $|c_n(\tau)|^2$ correspond to 'how much' of the quantum computer's state is in the nth Liouville space decay mode.

We now make two crucial assumptions: First, we assume that the quantum computer gates represented by $\mathcal{L}(t)$ do not take amplitude out of whatever subspace we are using and that our initial state $|\rho_0\rangle\rangle$ is located entirely within our subspace. This then implies the normalization condition $\sum_{n \in S} |c_n(\tau)|^2 = 1$ where the sum is over states $|\psi_n(\tau)\rangle\rangle$ which lie in our subspace. Secondly, we assume that the evolution of our quantum computer is sufficiently complex so that the time average $\frac{1}{t}\int_0^t |c_n(\tau)|^2 d\tau$ can be replaced by $\frac{1}{N_S}$ where N_S is the dimension of our subspace. This then leads to $F(t) = e^{-\overline{\gamma} t}$ where $\overline{\gamma} = \frac{1}{N_S} \sum_{n \in S} \gamma_n$ is the average decoherence rate in our subspace. Physically, the replacement of the time average over $|c_n(\tau)|^2$ by $\frac{1}{N_S}$ means that the operations of the quantum computer transfer amplitude ergodically through Liouville space such that each of the decay modes of Γ are equally sampled.

The above argument has hopefully demonstrated the plausibility of using an average decoherence rate as a measure of overall decoherence time. In this work we will not consider the issue further and now turn to the calculation of average decoherence rates in the presence of imperfectly correlated fields.

V. DECOHERENCE RATE SCALING DUE TO PARTIALLY CORRELATED FIELDS

In what follows, it will be useful to make a number of simplifying assumptions about the spectral densities

which appear in the relaxation superoperator Γ given in Eq.(3). First, we will assume that the qubits all have identical gyromagnetic ratios and that the root mean square field seen by each qubit is the same. This assumption is in no way essential and will merely serve to simplify our results by allowing us to express the spectral densities as $J^{ij}_{qq'}(\omega) = C_{ij}J_{qq'}(\omega)$ where the correlation coefficient C_{ij} is 1 for $i = j$. For $i \neq j$, C_{ij} is some number between 0 and 1 which represents the degree of correlation between the fields seen by the ith and jth qubits. Negative correlation coefficients corresponding to anti-correlated fields are of course also possible but we will not consider this case. Finally, we assume that there exists no correlation between the z component of the random field and the x and y components. With these assumptions we can now write the relaxation superoperator in the simplified form,

$$\Gamma\rho_{avg} = \frac{1}{2}J_{zz}(0)\sum_{i,j}C_{ij}[\sigma_z^{(i)},[\sigma_z^{(j)},\rho_{avg}]] + \frac{1}{2}J_{+-}(\omega)\sum_{i,j}C_{ij}\left([\sigma_+^{(i)},[\sigma_-^{(j)},\rho_{avg}]] + [\sigma_-^{(i)},[\sigma_+^{(j)},\rho_{avg}]]\right) \tag{13}$$

where ω is the Larmor frequency of the qubits which are all equal now due to the assumption of identical gyromagnetic ratios.

In order to evaluate the average decoherence rate in a given subspace, we simply need to trace Γ over the states in the subspace. If our subspace has d dimensions in Hilbert space, then we can write the average decoherence rate $\overline{\gamma}$ as,

$$\overline{\gamma} = \frac{1}{d^2}\sum_{n,m}\mathrm{Tr}[|\psi_n\rangle\langle\psi_m|\Gamma|\psi_m\rangle\langle\psi_n|] \tag{14}$$

where the $|\psi_n\rangle$ are any set of orthonormal basis states for our subspace. We will consider the average decoherence rate in subspaces where all matrix elements of the cluster operators given in Eq. (7) vanish. As discussed before, these subspaces are decoherence free in the limit where all qubits in each cluster see exactly the same field fluctuations.

We now derive two simple identities which will be useful in evaluating Eq. (14). First, we show that,

$$\sum_n\langle\psi_n|\sigma_z^{(i)}\sigma_z^{(j)}|\psi_n\rangle = \frac{-d_c}{n_c-1} \qquad (i \neq j) \tag{15}$$

where $i \neq j$ label two qubits in a cluster of n_c physical qubits and the sum runs over the d_c states $|\psi_n\rangle$ which are annihilated by the collective operator $\sigma_z^c = \sum_{i\in c}\sigma_z^{(i)}$ for the cluster. In this case, d_c is given by $\frac{n_c!}{(n_c/2)!^2}$ but for reasons of generality we will continue to write d_c. We begin by noting that σ_z^c commutes with all permutation operators which exchange two qubits in the cluster. Thus $\sum_n\langle\psi_n|\sigma_z^{(i)}\sigma_z^{(j)}|\psi_n\rangle$ is independent of the indices i and j for $i \neq j$ and we can write,

$$\begin{aligned}\sum_n\langle\psi_n|\sigma_z^{(i)}\sigma_z^{(j)}|\psi_n\rangle &= \frac{1}{n_c(n_c-1)}\sum_{i\neq j}\sum_n\langle\psi_n|\sigma_z^{(i)}\sigma_z^{(j)}|\psi_n\rangle \\ &= \frac{1}{n_c(n_c-1)}\left(\sum_n\langle\psi_n|\sigma_z^c\sigma_z^c|\psi_n\rangle - \sum_i\sum_n\langle\psi_n|(\sigma_z^{(i)})^2|\psi_n\rangle\right)\end{aligned} \tag{16}$$

from which Eq. (15) follows by first noting that $\sum_n\langle\psi_n|\sigma_z^c\sigma_z^c|\psi_n\rangle$ is equal to zero since the states $|\psi_n\rangle$ are annihilated by the collective operators σ_z^c and then that $\sum_i\sum_n\langle\psi_n|(\sigma_z^{(i)})^2|\psi_n\rangle = n_c d_c$ since $(\sigma_z^{(i)})^2$ is equivalent to the identity operator. We also have the identity,

$$\sum_n\langle\psi_n|\sigma_z^{(i)}|\psi_n\rangle = 0 \tag{17}$$

which can be shown by an argument similar to that given above.

We now return to the evaluation of the average decoherence rate $\overline{\gamma}$ given in Eq. (14). For simplicity, we will first consider the case of pure dephasing for a cluster of n_c qubits where Γ is given simply by,

$$\Gamma\rho_{avg} = \frac{1}{2}J_{zz}(0)\sum_{i,j}C_{ij}[\sigma_z^{(i)},[\sigma_z^{(j)},\rho_{avg}]] \tag{18}$$

If the qubits all saw exactly the same field fluctuations then the subspace of states annihilated by the collective

cluster operator $\sigma_z^c = \sum_{i\in c}\sigma_z^{(i)}$ would be decoherence free. We will refer to this subspace as the $m_z = 0$ subspace. Armed with the identities given in Eqs. (15) and (17) it is a simple matter to show that in the $m_z = 0$ subspace, Eq. (14) yields an average decoherence rate of,

$$\overline{\gamma} = n_c J_{zz}(0)(1 - C_{avg}) \tag{19}$$

where C_{avg} is the average cross correlation coefficient given by,

$$C_{avg} = \frac{1}{n_c(n_c-1)}\sum_{i\neq j} C_{ij} \tag{20}$$

Since a Hilbert space of dimension d_c can be used to encode $n_{LQ} = \mathrm{Log}_2 d_c$ logical qubits, we can write the average decoherence rate per logical qubit as,

$$\overline{\gamma}/n_{LQ} = \frac{n_c}{\mathrm{Log}_2 d_c} J_{zz}(0)(1 - C_{avg}) \tag{21}$$

A scalable quantum computer would then be constructed by concatenating many of these clusterized subspaces together to create basis states of the form $|\psi_n\rangle \otimes |\psi'_{n'}\rangle \otimes \cdots$ where the primes indicate that $|\psi_n\rangle$ and $|\psi'_{n'}\rangle$ are the basis states for separate clusters of qubits. Using Eqs. (17) and (14) one can show that the average decoherence rate is unaffected by correlations in the fields seen by different clusters. Thus, Γ can be written as a sum of terms, each of which acts on a single cluster of qubits and the average decoherence rate therefore scales linearly with the number of clusters. Furthermore, the number of logical qubits which we can encode also scales linearly with the number of clusters and we therefore have that Eq. (21) is valid not only for a single cluster of physical qubits but also for an arbitrary number of such clusters used together to create a quantum computer. Simply put, we have shown that for a clusterized encoding where logical qubits are encoded in the states annihilated by the collective cluster operators $\sigma_z^c = \sum_{i\in c}\sigma_z^{(i)}$, decoherence rates scale linearly with the number of logical qubits with a proportionality constant given in Eq. (21).

Examining Eq. (21) which we now understand to represent decoherence rate scaling, we see that it contains the factor $\frac{n_c}{\mathrm{Log}_2 d_c}$ which is nothing more than one over the efficiency of our encoding. Now on the one hand, larger clusters lead to higher encoding efficiencies [6] and thereby to an *decrease* of the factor $\frac{n_c}{\mathrm{Log}_2 d_c}$ but in order to determine the effect of cluster size on decoherence rate scaling, we must also consider the factor $(1-C_{avg})$ which we generally expect to *increase* for larger clusters.

In order to further investigate the dependence of decoherence rate scaling on cluster geometry, it will be useful to specify a model for the C_{ij}. For the sake of simplicity, we consider only the two cases where the correlation coefficients have either an exponential or gaussian dependence on the physical separation between the ith and jth qubits so that we have either $C_{ij} = e^{-\alpha r_{ij}}$ or $C_{ij} = e^{-\beta r_{ij}^2}$. In the limit where the intra-cluster correlation coefficients are all close to 1, we can then write the factor $(1 - C_{avg})$ as,

$$1 - C_{avg} \cong \alpha\overline{r} \tag{22}$$

for the case of exponentially decaying spatial correlations or

$$1 - C_{avg} \cong \beta\overline{r^2} \tag{23}$$

for the case of gaussian decaying spatial correlations where the bar indicates an average over qubit pairs. Combining this with Eq. (21) gives the decoherence scaling rate as $\frac{n_c}{\mathrm{Log}_2 d_c} J_{zz}(0)\alpha\overline{r}$ or $\frac{n_c}{\mathrm{Log}_2 d_c} J_{zz}(0)\beta\overline{r^2}$ for the case of exponential or gaussian spatial field correlations respectively. Written in this form, one can see clearly that the beneficial effect of increasing the encoding efficiency ($\frac{\mathrm{Log}_2 d_c}{n_c}$) by using larger cluster sizes must be weighed against the accompanying increase in the average inter-qubit separation. Physically this simply reflects the fact that the random field fluctuations are more strongly correlated over shorter distances which is what led us to consider the clusterized encodings in the first place.

In table (I) we evaluate the above expressions for decoherence rate scaling for a number of cluster geometries. The physical arrangement of the qubits in the clusters are chosen so that all geometries have the same nearest neighbor inter-qubit separation. For the $m_z = 0$ subspaces, which protect only against pure dephasing, the results are expressed in terms of the rate scaling that is obtained for the pairwise encoding, and we see that the tetrahedral and octahedral geometries that allow for a low average inter-qubit separation give rise to the best scaling. The 2x2 and 2x3 square encodings on the other hand do not utilize three dimensional space as efficiently as the tetrahedral and octahedral geometries do. For the case of gaussian spatially correlated fields which give a higher penalty for increased inter-qubit separation, these encodings are outperformed by the simple pairwise encoding.

It would be nice to make a general statement as to which cluster sizes or geometries give rise to the most favorable decoherence rate scaling. This is, however, complicated by the fact that the encoding efficiency as a function of cluster size varies depending on the type of encoding. We can, for example, consider subspaces spanned by the singlet states which are annihilated not only by the collective cluster operators σ_z^c, but also by the collective raising and lowering operators $\sigma_\pm^c = \sum_{i\in c}\sigma_\pm^{(i)}$. These singlet subspaces therefore protect not only against pure dephasing but also against amplitude damping induced by T_1 processes [5, 12]. By carrying out an analysis similar to that in Eqs. (15-20), one can show that the singlet subspaces give rise to average decoherence rates per logical qubit of,

$$\overline{\gamma}/n_{LQ} = \frac{n_c}{\mathrm{Log}_2 d_c}(J_{zz}(0) + 2J_{+-}(\omega))(1 - C_{avg}) \tag{24}$$

TABLE I: Comparison of decoherence rate scaling for different cluster geometries using the exponential and gaussian models for the spatial dependence of the field correlations. For the $m_z = 0$ subspaces, rates are listed in terms of the rate obtained for the pairwise encoding, for the singlet subspaces, rates are in terms of that obtained for the tetrahedral encoding.

Geometry	# Phys.	$m_z = 0$		Singlet	
	Qubits	Exp	Gauss	Exp	Gauss
Pair	2	1.00	1.00		
Tetrahedral	4	0.77	0.77	1.00	1.00
Square 2x2	4	0.88	1.03	1.14	1.33
Square 2x3	6	0.98	1.53	0.91	1.42
Octahedral	6	0.75	0.83	0.70	0.78
Cubic 2x2x2	8	0.83	1.12	0.67	0.90
Cubic 2x3x3	18	1.01	1.94	0.64	1.23
Cubic 4x4x4	64	1.37	4.02	0.75	2.19

which is very similar to the rate scaling that we derived earlier except that the dimension of the singlet subspace for a cluster of n_c qubits is not equal to $\frac{n_c!}{(n_c/2)!^2}$ but is instead given by $n_c![(n_c/2)!(n_c/2+1)!]^{-1}$ [5]. In table (I) we also compare the decoherence rate scaling of the singlet subspaces for a number of cluster sizes and geometries. Since a minimum of four physical qubits are needed to encode one logical qubit, we give the decoherence rate scaling factors for the singlet subspaces in terms of the rate obtained for a cluster of four physical qubits in a tetrahedral geometry.

Again examining the results in table (I), we see that of the clusterizations considered, the cubic 2x3x3 cluster with 18 physical qubits gives the best rate scaling when the random field fluctuations have an exponential spatial correlation whereas the octahedral cluster with 6 physical qubits wins out if the fields have a gaussian spatial correlation. These results reflect the fact that the efficiency of encoding using the singlet states of 18 physical qubits ($\frac{1}{18}\mathrm{Log}_2\frac{18!}{9!10!} = .68$) represents a nearly three-fold increase over that obtained by using only four qubits ($\frac{1}{4}\mathrm{Log}_2\frac{4!}{2!3!} = .25$). For the case of the $m_z = 0$ encodings, the increased efficiency obtained upon going from a cluster of four qubits to a cluster of 18 qubits is not nearly so great and the rate scaling is actually slightly worse than one obtains using pairs of physical qubits. Thus, we see that which cluster geometry gives the best decoherence rate scaling depends both upon the type of encoding used and also the form of the spatial field correlations. Note also that because we have made the assumption of strong spatial field correlations in Eqns. (22) and (23), the results in table (I) do not depend upon the actual amount of spatial field correlation but only upon the form of the spatial correlations as a function of distance. In other words, in the limit of strong correlation, the optimal cluster size does not depend upon how small the parameters α or β are in the expressions $C_{ij} = e^{-\alpha r_{ij}}$ and $C_{ij} = e^{-\beta r_{ij}^2}$, but only on the functional form of the C_{ij} as a function of inter-qubit separation.

VI. CONCLUSION

We have analyzed the scaling of average decoherence rates with system size and have seen that in order to obtain an encoding which optimizes decoherence rate scaling, one must find a cluster size and geometry which balances encoding efficiency against the degree of correlation in the random field fluctuations seen by the physical qubits.

Note that we have made no attempt to define in detail how individual logical qubits should be encoded, but have only considered how decoherence rates scale for subspaces which could in principle be used to encode qubits. Except for the pairwise clustering, all of the encodings considered above would require at least some of the logical qubits to be defined in terms of states involving entanglements between physical qubits from separate clusters which of course further complicates the already difficult task of designing gates. Nevertheless, one can still obtain improved scaling rates even without using all of the sub-decoherent states of the individual clusters. For example, a 2x2x2 cubic cluster of physical qubits yields 14 singlet states and even if we only use 8 of these to encode 3 qubits, the encoding still gives a scaling rate which is improved over that for a tetrahedral geometry with 4 physical qubits. Moreover, the logical qubits are now defined such that single qubit gates can be implemented by manipulating single clusters of physical qubits.

As an additional possibility, one could also consider how scaling rates are affected by incorporating the use of states from subspaces which are not decoherence free in the limit of perfect collective decoherence but which are still sub-decoherent in the presence of spatially correlated field fluctuations. Whatever the details of the encoding, however, we expect that the principle of optimizing decoherence rate scaling by balancing the encoding efficiency against the average spatial correlations of the bath will remain important.

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